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Asymmetric private values auctions with application to joint bidding and mergers

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Abstract

Much of the theoretical auction literature assumes that there are no observable differences between the bidders of an auction. In this paper, I present an asymmetric auction model where the distribution of a bidder's value depends on some commonly observed characteristics. The model is applied to joint bidding and mergers in auction markets. The results provide additional evidence that first-price auctions are less susceptible to the acquisition and exercise of market power than open auctions.

Keywords: Auction; Bidding; Mergers

JEL classification: D44; L40

1. Introduction

A number of applications of auction models requires the analysis of asymmetric auctions. Most notable among these are the analyses of joint bidding, mergers, and collusive behavior. Yet there is relatively little literature within the theory of auctions on the subject of first-price auctions with asymmetric bidders.[1] Thus,

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[1] For a survey of the economic literature on auctions see McAfee and McMillan (1987). Also see Hendricks and Porter (1989) for a comprehensive survey of the literature on collusive behavior in auctions.

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there are a number of basic questions relating to the acquisition and application of market power in bidding markets that, as of yet, remain unanswered. In this paper, I present some results for auctions with asymmetric bidders as applied to joint bidding and mergers.

While the analysis of first-price auctions with asymmetric bidders involves the difficult task of analyzing a system of differential equations, the equilibrium strategy in second-price asymmetric auctions is relatively simple to obtain. The equilibrium bidding strategy in both symmetric and asymmetric second-price auctions is for each bidder to bid his value for the object at auction. Thus, a number of researchers (Graham and Marshall, 1987; von Ungern-Sternberg, 1988; Mailath and Zemsky, 1991) have been able to analyze certain applications of asymmetric second-price auctions. As was initially described by Robinson (1985), they find that open and ascending value oral auctions are particularly susceptible to the formation of bidding rings or cartels. Assuming that a ring is able to select the member with the highest value for the object at auction to bid without competition from the other members, the other members of the ring have no incentive to defect from the ring by competing with the ring-sanctioned bidder. This result has an important implication when applied to mergers. Potential entry into a market is believed to limit a firm's ability to acquire undue market power through merger or collusive behavior. However, mergers or the formation of collusive rings have no effect on the profits of firms remaining outside of the merger or ring (assuming constant returns-to-scale) and, thus, mergers and collusion provide no additional incentive for entry into the market where price is determined by second-price auction.

There have been a few models of first-price auctions with asymmetric bidders. Maskin and Riley (1985), McAfee and McMillan (1992), and Hausch (1987) analyze bidders with binomially distributed values. Dalkir (1991), Lebrun (1995), Marshall et al. (1994), Maskin and Riley (1992a), and Pesendorfer (1996) have analyzed bidders with continuously distributed values. Dalkir (1991) has shown the existence of increasing and differentiable equilibrium bidding strategies for a certain class of distributions. Lebrun (1996) shows the existence of Nash equilibrium bidding strategies in a more general setting. The numerical analyses of Dalkir (1991) and Marshall et al. (1994) are consistent with the analytical results demonstrated here. Maskin and Riley (1992a) analyze two bidder asymmetric first-price auctions.

In this paper, I present an auction model in which bidders have different characteristics affecting the distribution of their values. Each bidder's characteristics are observed by the other bidders. A bidder's characteristics provide information regarding the bidder's value to the other bidders. In applying the model to joint bidding, the size of a bidding ring or merged firm provides the other bidders with information regarding the ring's value for the object at auction. The larger the size, the higher the value the ring is expected to have. Assuming that the equilibrium is well-behaved, this structure allows me to derive a number of qualitative results regarding the bidding strategies, probabilities of winning, and expected payoffs of the bidders. These results generalize some of the results of Maskin and Riley (1992a) to bidding markets with more than two bidders.

The model is easily applied to the cases of joint bidding and mergers. DeBrock and Smith (1983), Feinstein et al. (1985), Marshall et al. (1994), and Pesendorfer (1996) have analyzed similar models of joint bidding.[2] In these applications, the type of a group of individuals bidding together is taken to be the size of the group. For first-price auctions, the model indicates that the per member expected payoff of smaller groups is greater than the per member expected payoff of larger groups. This result implies that in first-price auctions profitable mergers increase the payoffs of nonmerging bidders and potential entrants. This result is in contrast to the effects of a merger in second-price auctions where mergers have no effect on the payoffs of nonmerging bidders or potential entrants.

2. The model

Consider a first-price auction for the right to purchase an indivisible object. Let $N=\{1,\ldots,n\}$ denote the set of bidders. In a first-price auction the highest bidder wins the auction and pays a price equal to his bid. In the case of multiple winning bids, the winner is selected randomly so that each bidder submitting the maximum bid has the same probability of winning. Associated with each bidder $i \in N$ is a random variable V_i and a type t_i. The realization of V_i is bidder i's *value* for the object at auction and is known to bidder i but unknown to the other bidders. Bidder i's type t_i is common knowledge, and it provides the other bidders with information about i's value for the object at sale. Hence, a bidder's beliefs about his rivals' values depends on the type of each of his rivals. In the standard symmetric auction model all bidders have the same type, and hence, there is no scope for a bidder to have different beliefs regarding the values of different rivals.

Higher types are associated with higher values for the object at auction. Hence, a bidder's type can be thought of as representing an observable characteristic of the bidder that is associated with higher values for the object for sale. Assuming economies-of-scale in values, one could interpret that characteristic as the bidder's size or scale. For instance, after the merger of two bidders, the new bidder is larger than either of the original two bidders. Other bidders will believe that this new, larger bidder is more likely to have a higher value than either of the original two bidders. A bidder's type alternatively could represent the bidder's distance from the market.[3] Consider an example of saw mills bidding for the right to cut timber on federal lands. The location of each bidder's mill is likely to be known by the other bidders, and the closer a bidder's mill is to the timber tract up for auction the

[2] See Rothkopf and Engelbrecht-Wiggans (1992) for a critique of DeBrock and Smith (1983).

[3] To make this interpretation consistent with the model, bidders closer to the market would have higher types.

higher his value for that timber is likely to be. Hence, distance is an observed characteristic of each bidder that is associated with the bidders' values for the object at auction.

Assume that conditioned on t_i the distribution of V_i is independent of V_j for all $j \in N \backslash \{i\}$. Let $\mathscr{V} = [\underline{v}, \bar{v}]$ denote the support of V_i, and let $T \subset \mathbb{R}$ denote the set of types. Let $F(\cdot|t_i)$ denote the distribution function of V_i conditional on the bidder's type t_i for all i. The density $f(v|t_i)$ exists and is positive for all $v \in (\underline{v}, \bar{v})$ and $t_i \in T$. I make the following assumption regarding the conditional density f.

Strict Multivariate Total Positivity of Order 2 (SMTP$_2$): f satisfies SMTP$_2$ if

$$f(v|t)f(v'|t') > f(v|t')f(v'|t) \quad (1)$$

for all $v, v' \in (\underline{v}, \bar{v})$ such that $v > v'$ and for all $t, t' \in T$ such that $t > t'$.

SMTP$_2$ is a strict version of the *Multivariate Total Positivity of Order 2* property described by Karlin and Rinott (1980). SMTP$_2$ is also equivalent to the strict monotone likelihood ratio property described by Milgrom (1981). This property is stronger than the property of affiliation described by Milgrom and Weber (1982). A basic property of SMTP$_2$ functions (shown by Karlin and Rinott, 1980) is that

$$E[\phi(V_i,t_i)|t_i = t] > E[\phi(V_i,t_i)|t_i = t'], \ \forall t > t', \quad (2)$$

whenever ϕ is an increasing function.[4] Thus, a bidder with a higher type is expected to have a higher value for the object at auction. Another important result of SMTP$_2$ density functions is that

$$\frac{f(v|t)}{F(v|t)} > \frac{f(v|t')}{F(v|t')},$$

for all $t > t'$ and for all $v \in (\underline{v}, \bar{v})$.

At the time of the bidding, each bidder's type is common knowledge. A bidder's beliefs about the values of bidders of different types t and t' are defined by the distributions $F(\cdot|t)$ and $F(\cdot|t')$. Since these distributions are different, the model described involves bidding by asymmetric bidders (commonly referred to as asymmetric auctions). Clearly, the symmetric model is a special case of this model, corresponding to the situation where the types of all of the bidders are equal. In fact, when all of the bidders are of the same type, the informational environment is exactly that of the independent private values environment.

Before going on to describe the features of first-price auctions in this environment, it is of interest to analyze second-price and open auctions. In a second-price auction the good is sold to the highest bidder at a price equal to the second highest bid. Open auctions are strategically equivalent to second-price

[4] Karlin and Rinott (1980) formulate this property with a weak inequality. However, with the strict version of SMTP$_2$ defined above, the inequality is strict.

auctions if the values of the bidders are stochastically independent. Hereafter, I use "open auctions" to refer to ascending value oral auctions. The equilibria for second-price and open auctions are unaffected by the asymmetry of bidders. Just as in the symmetric case, the dominant strategy for a bidder is to bid his value.[5] To see this, fix any set of competing bids. If the bidder does not win by bidding his value then lowering his bid will not affect his payoff. Raising his bid so that he wins the auction would force him to purchase the good at a price above his value. If the bidder does win by bidding his value, he receives a non-negative payoff. Raising the bid will have no effect on his payoff. Lowering the bid only has an effect if the bid is lowered below the highest competing bid in which case the bidder loses the auction and receives a zero payoff.

Moving on to first-price auctions, I assume that the set of bidder types $T=\{t_1,\dots,t_n\}$ is fixed. To simplify the notation, let $F^i(\cdot)=F(\cdot|t_i)$, for all $i\in N$. Similarly, let f^i denote the density of F^i. Given the set of types T, a pure bidding strategy for i is a function $\beta^i:\mathcal{V}\rightarrow\mathbb{R}$. I make the assumption that the equilibrium bidding function β^i is increasing and differentiable in v.

Suppose that each bidder $j\in N\backslash\{i\}$ uses a strategy β^j that is increasing and differentiable. Then the expected payoff of bidder i given that he bids b and has value v can be written as

$$\pi^i(b,v)=[v-b]\prod_{j\in N\backslash\{i\}}p^j(b),$$

where $p^j(b)$ denotes the probability that bidder j bids below b. Thus, $p^j(b)$ is the distribution function of bidder j's bids conditional on the type t_j. For each bidder $i\in N$ define the function $z^i:\mathbb{R}\rightarrow\mathbb{R}$ such that $z^i(\beta^i(v))=v$. For $b<\beta^i(\underline{v})$, let $z^i(b)=\underline{v}$, and for $b>\beta^i(\bar{v})$, let $z^i(b)=\bar{v}$. On the range of β^i, z^i is the inverse of β^i. Thus, p^i can be written as $p^i(b)=F^i(z^i(b))$.

Lemma 1. *If $(\beta^1,\dots,\beta^n)$ is an increasing and differentiable equilibrium strategy profile in a first-price auction, then for all $i,j\in N$,*

(i) $\beta^i(\underline{v})=\underline{v}$,
(ii) $\beta^i(\bar{v})=\beta^j(\bar{v})$,
(iii) $\pi^i_1(\beta^i(v),v)=0$, for all $v\in\mathcal{V}$, where $\pi^i_1(\cdot,\cdot)$ denotes the partial derivative of π^i with respect to its first argument.

(See Appendix A for the proof.)

Notice that Lemma 1 does not require the distributions of values to satisfy

[5] In open auctions the price starts low and is raised until there is only one active bidder left. A bidding strategy in this environment corresponds to the price at which the bidder will drop out of the bidding.

$SMTP_2$. These conditions are identical to those shown by Dalkir (1991) for a special case of this model. These conditions also generalize the necessary conditions for the equilibrium found by Maskin and Riley (1992a) for the two bidder case. Parts (i) and (iii) of Lemma 1 arise as necessary conditions for symmetric equilibrium bidding strategies. However, it is less clear why part (ii) should hold for asymmetric auctions.

While the complete proof is in Appendix A, I present a sketch of the proof here to provide some indication of why this condition is necessary. If (ii) does not hold, then there must be at least one bidder i whose probability of winning with his bid $\beta^i(\bar{v})$ is equal to one, and there must be at least one bidder j whose probability of winning with his bid $\beta^j(\bar{v})$ is less than one. Thus, the bid $\beta^i(\bar{v})$ always wins, and the bid $\beta^j(\bar{v})$ only sometimes wins. Since bidder j would also always win if he submitted $\beta^i(\bar{v})$, it must be that

$$\pi^i(\beta^i(\bar{v}),\bar{v}) = \pi^j(\beta^i(\bar{v}),\bar{v}). \tag{3}$$

Furthermore, bidders i and j's probabilities of winning with the bid b can be written as $[\prod_{k\in N} p^k(b)]/p^i(b)$ and $[\prod_{k\in N} p^k(b)]/p^j(b)$, respectively. Notice that bidder i has a higher probability of winning with the bid $\beta^j(\bar{v})$ than bidder j since $p^j(\beta^j(\bar{v}))=1$ and $p^i(\beta^j(\bar{v}))<1$. Thus, it is straightforward to see that

$$\pi^i(\beta^j(\bar{v}),\bar{v}) > \pi^j(\beta^j(\bar{v}),\bar{v}). \tag{4}$$

However, Eqs. (3) and (4) are inconsistent with equilibrium which requires that

$$\pi^i(\beta^i(\bar{v}),\bar{v}) \geq \pi^i(\beta^j(\bar{v}),\bar{v}),$$

$$\pi^j(\beta^j(\bar{v}),\bar{v}) \geq \pi^j(\beta^i(\bar{v}),\bar{v}).$$

While there is no general existence proof of increasing and differentiable equilibria under the assumptions made thus far, it is known that an increasing and differentiable equilibrium set of bidding strategies exists in the symmetric case–when all bidders are of the same type. Maskin and Riley (1992b) prove that in general an equilibrium exists in asymmetric auctions without demonstrating that the strategies are differentiable. In addition, Dalkir (1991) has shown the existence of equilibrium bidding strategies that are increasing and differentiable and conditions (i), (ii), and (iii) of Lemma 1 for a special case of the model. He analyzes a two bidder model equivalent to the special case where $F(v|t_i)=G(v)^{t_i}$, t_1 and t_2 are integers, and G is the uniform cumulative distribution function. It is easy to verify that $G(v)^t$ satisfies $SMTP_2$.[6]

[6] In Section 5, I demonstrate that this is indeed a special case of the model described here (see footnote 10).

3. Bidding behavior in asymmetric auctions

The following proposition presents a number of properties of equilibrium bidding behavior in asymmetric first-price auctions. Let $\bar{b}$ denote the bid at $\bar{v}$ for all the bidders (i.e., $\bar{b}=\beta^i(\bar{v})$, for all $i \in N$).

Proposition 1. *Suppose $(\beta^1,\ldots,\beta^n)$ is an increasing and differentiable equilibrium strategy profile for a first-price auction and the conditional density function $f(\cdot|\cdot)$ satisfies $SMTP_2$. Then for all $i,j \in N$ such that $t_i < t_j$,*

(i) $\beta^i(v) > \beta^j(v)$, for all $v \in (\underline{v},\bar{v})$,
(ii) bidder j has a higher probability of winning the auction than bidder i,
(iii) the distribution of bidder j's bids stochastically dominates the distribution of bidder i's bids; that is, $p^i(b) > p^j(b)$, for all $b \in (\underline{v},\bar{b})$,
(iv) bidder j has a higher expected payoff than bidder i.

Furthermore, if $(\beta^1,\ldots,\beta^n)$ is an increasing and differentiable equilibrium strategy profile to a second-price or open auction and the conditional density function $f(\cdot/\cdot)$ satisfies $SMTP_2$, then parts (ii)–(iv) hold for all $i,j \in N$ such that $t_i < t_j$.

(See Appendix A for the proof.)

Maskin and Riley (1992a) show that parts (i) and (iii) of Proposition 1 hold for markets with two bidders.[7]

Part (i) of Proposition 1 implies that bidders of lower types bid more aggressively than bidders of higher types. That is, for a given value, lower types submit a bid closer to their value than higher types. However, even though lower types bid more aggressively, part (ii) indicates that higher types bid higher on average than lower types.

These results have some interesting implications for asymmetric auctions. It is evident from part (i) of Proposition 1 that the allocation resulting from an asymmetric first-price auction may be inefficient.[8] When bidders are symmetric and each follows the same pure strategy, then the bidder with the highest value always wins the auction. However, in a first-price asymmetric auction the bidder with the highest value does not always win the auction. In contrast, allocations from second-price and open auctions are efficient even when bidders' values are not identically distributed. In second-price and open auctions each bidder bids his value, and thus, the bidder with the highest value always wins.

[7] Independent of my research, Lebrun (1995) has shown similar results for the n bidder case.
[8] This fact was first pointed out by Maskin and Riley (1985).

4. Application to joint bidding and mergers

Consider a situation where there are n bidders. Each bidder $i=1,\ldots,n$ receives k_i independent draws from the random variable X with distribution G. Furthermore, if bidder i's draws from X are $x_1,\ldots,x_{k_i}$, then bidder i's value is $v_i=\max\{x_1,\ldots,x_{k_i}\}$.

Such a model can be used to describe auctions with joint bidding or mergers.[9] Interpret each bidder i as a group of k_i symmetric individuals. Each individual's value for the object of the auction is equal to his draw from X. Thus, if bidding group i has k_i members, then the group as a whole has k draws from X. An implicit assumption of the model is that each group efficiently allocates any gains from trade among its members. If a winning group efficiently allocates the good among themselves, then the member with the highest value must receive the good, and thus, the group's value for the object of the auction is the highest of the values of its members. McAfee and McMillan (1992) discuss a number of ways that bidding rings can achieve an efficient allocation among themselves. Their basic result is that in first-price auction markets the contract binding the group must be enforceable if the allocation is to be efficient. Such enforcement is possible when the group is legally sanctioned as in the case of mergers or allowed joint bidding. However, enforcement would be more difficult in the case of illicit combinations such as bidding rings.

Under some additional assumptions this setup is a special case of the model described above. Let the type associated with group i be the number of individuals jointly bidding as group i. That is, $t_i=k_i$, and, thus, $F(v|t_i)=G(v)^{t_i}$.[10] F can be represented in this manner as long as membership in a group is independent of an individual's value. If G is strictly increasing for all $v\in\mathcal{V}$, then SMTP$_2$ is satisfied. To see this let $g(v)=G'(v)$ and, thus, $f(v|t)=tG(v)^{t-1}g(v)$. Substituting this expression into Eq. (1) yields

$$tG(v)^{t-1}g(v)t'G(v')^{t'-1}g(v')>t'G(v)^{t'-1}g(v)tG(v')^{t-1}g(v')$$

for $v\in(\underline{v},\bar{v})$ and $t\in T$ such that $v>v'$ and $t>t'$. Notice that the inequality holds if and only if

$$G(v)^{t-1}G(v')^{t'-1}>G(v)^{t'-1}G(v')^{t-1}.$$

Rearranging this inequality yields $G(v)^{t-t'}>G(v')^{t-t'}$, which holds since $t>t'$ and $G(v)>G(v')$.

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[9] McAfee and McMillan (1992) and Dalkir (1991) use this type of setup in the analysis of bidding rings and mergers, respectively. Marshall et al. (1994) numerically analyze the same model where G is uniformly distributed. The empirical analysis of Pesendorfer (1996) indicates that the results from such models are consistent with alleged cartel behavior in auctions for school milk contracts in Florida and Texas.

[10] Notice that if G is the uniform distribution, this is the distributional assumption for which Dalkir (1991) shows the existence of increasing and differentiable equilibria in the two bidder case.

As a special case of the asymmetric model described in Section 2, the results of Proposition 1 apply to the present context. Given a particular value, smaller groups bid more aggressively than larger groups (part (i) of Proposition 1). That is, given a particular value smaller groups bid higher than larger groups. In addition, by Proposition 1 larger groups win more often than smaller groups, and the distribution of the bids of larger groups stochastically dominates the distribution of bids from smaller groups. These last two results make the model straightforward to test empirically as long as the bids and the size of the groups are observable.[11] Proposition 1 also implies that larger groups have a higher expected group payoff than smaller groups.

One way in which to look at the effects of a merger is to analyze the per member payoff of a bidding group. Define U^i as the per member expected payoff of group i. That is,

$$U^i = \frac{1}{t_i}\int_{\underline{v}}^{\bar{v}} \pi^i(\beta^i(v),v)f^i(v)\,dv,$$

where $\pi^i(b,v)$ is defined as the expected payoff of bidder i when the other bidders follow their equilibrium bidding strategy and bidder i has value v and submits bid b.

Proposition 2. *Suppose $(\beta^1,\ldots,\beta^n)$ is an increasing and differentiable equilibrium strategy profile and $F(v|t) = G(v)^t$. Then in a first-price auction $U^i > U^j$ and in a second-price or open auction $U^i < U^j$, for all $i,j \in N$ such that $t_i < t_j$.*

Proof. I begin by showing the part of the Proposition relating to a first-price auction. Notice that for a first-price auction U^i can be rewritten as

$$U^i = \int_{\underline{v}}^{\bar{v}} [v - \beta^i(v)] \prod_{h\in N\setminus\{i,j\}} p^h(\beta^i(v))G(z^j(\beta^i(v)))^{t_j}G(v)^{t_i-1}g(v)\,dv. \tag{5}$$

Extend the notation so that $U^i(\beta^i,\beta^j)$ is group i's per member expected payoff when group i bids according to β^i and group j bids according to β^j. Similarly, let $U^j(\beta^i,\beta^j)$ denote group j's per member expected payoff when group i bids according to β^i and group j bids according to β^j. Then

$$U^i \geq U^i(\beta^j,\beta^j) = U^j(\beta^j,\beta^j) > U^j.$$

The first weak inequality follows because if β^i is an equilibrium, then for group i, bidding according to β^i must be at least as good as bidding according to β^j. The

[11] In fact, the empirical analysis of Pesendorfer (1996) includes just such a test.

equality follows from the fact that when group i bids according to β^j, Eq. (5) becomes

$$U^i(\beta^j,\beta^j) = \int_{\underline{v}}^{\bar{v}} [v - \beta^j(v)] \prod_{h\in\mathcal{N}\setminus\{i,j\}} p^h(\beta^j(v)) G(v)^{t_j+t_i-1} g(v)\, dv.$$

When a similar expression is written out for $U^j(\beta^j,\beta^j)$ it is easy to see that they are equal. The last inequality follows because $G(v) > G(z^i(\beta^j(v)))$ for all $v \in (\underline{v},\bar{v})$ which is a direct result of part (i) of Proposition 1.

Now consider the result for second-price and open auctions. For a second-price or open auction a bidder's payoff π^i can be written as

$$\pi^i(b,v) = \int_{\underline{v}}^{b} [v - \xi](\hat{t} - t_i) G(\xi)^{\hat{t}-t_i-1} g(\xi)\, d\xi,$$

where $\hat{t} = \Sigma_{j=1}^{n} t_j$. Integrating the expression above by parts, group i's expected payoff when it has value v and follows its equilibrium strategy can be written as

$$\pi^i(v,v) = \int_{\underline{v}}^{v} G(\xi)^{\hat{t}-t_i}\, d\xi.$$

The expected payoff per member is

$$U^i = \frac{1}{t_i} \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v} G(\xi)^{\hat{t}-t_i} t_i G(v)^{t_i-1} g(v)\, d\xi\, dv, \tag{6}$$

which is increasing in t_i, holding $\hat{t}$ constant. Thus, larger groups have a higher per member expected payoff. □

McAfee and McMillan (1992) derive the conclusion of Proposition 2 for the case where $\mathcal{V} = \{0,1\}$ and G is the distribution function of a Bernoulli random variable. Their model and the one analyzed here are not nested since neither is a special case of the other.

5. Discussion

For first-price auctions Proposition 2 implies that, fixing the total number of bidding groups, the per member payoff of bidding groups in a first-price auction is larger the fewer the members of a group. While Proposition 1 implies that the expected *group* payoff of a larger group is greater than that of a smaller group, the expected *per member* payoff of a smaller bidding group is greater than the per member payoff of a larger group. An individual with the ability to decide between joining a larger or smaller bidding group would opt to join the smaller group.

While the result presented in Proposition 2 allows us to compare the per member payoffs of the groups in a particular market, it does not provide a comparison of per member payoffs when the groupings in a market change. Therefore, the result does not rule out the possibility that two groups may have an incentive to merge even though such a merger would create a larger group.[12] As an example, consider a market with three individuals. Let each individual be the only member in a bidding group. Suppose that two individuals are considering a merger. The merger would create a market with two bidding groups: a one member group and a two member group. If the per member payoff before such a merger is lower than the per member payoff for the two member group after the merger, then while the participants in the merger are assumed to benefit from their consolidation, the nonparticipant would benefit even more. Before the merger, symmetry implies that the three individuals have the same expected payoff; Proposition 2 implies that after the merger, the nonparticipating individual would have a higher expected payoff than the two participating individuals. Therefore, the choice to merge exhibits a "free rider" aspect that may hinder mergers in first-price bidding markets even when an incentive for merger exists. In the example, all three individuals prefer to be the nonparticipant in a merger. Hence, each individual may delay agreeing to join a merger hoping that the other two individuals will merge instead. The increase in the per member payoffs of nonparticipating groups may also increase the incentive for new entry into the market further limiting the anti-competitive effects of a merger.

The results presented in Proposition 2 indicate that second-price and open auctions perform differently from first-price auctions. Notice from Eq. (6) that the per member expected payoff of a group in an open auction depends on its own size and $\hat{t}$, the total number of individuals in the market, but it does not depend on the grouping of the remaining $\hat{t} - t_i$ individuals who are not members of i. Thus, another implication of Eq. (6) is that if groups i and j decide to join together to form one group, then the per member payoff of all of the other groups is unaffected by this merger since $\hat{t}$, the total number of individuals, remains the same. Therefore, while merger decisions in first-price bidding markets exhibit a "free rider" aspect, that is not the case in second-price and open auction bidding markets. This result also implies if a group can credibly commit to exclude a new entrant from membership, then mergers among individuals already in a second-price or open auction bidding market have no effect on the payoff of a potential entrant. If new entrants can only form new groups rather than join existing bidding groups, then mergers do not provide any additional incentive for new entry.

[12] I say that two groups would have an incentive to merge if the per member payoff of the merged group is higher than the per member payoff of the two original bidding groups.

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Appendix A

Proof of Lemma 1. The proof of part (i) is completed by first showing that $\beta^i(\underline{v}) = \beta^j(\underline{v})$, for all $i,j \in N$. This is shown by deriving a contradiction. Let $i = \arg\max_{k\in N}\{\beta^k(\underline{v})\}$. Suppose that there exists $j \in N$ such that $\beta^i(\underline{v}) > \beta^j(\underline{v})$. By the continuity of the bidding strategies implied by their differentiability and monotonicity, there exists a $\tilde{v} > \underline{v}$ such that $\beta^j(\tilde{v}) < \beta^i(\underline{v})$. Since i's bid is always greater than $\beta^j(\tilde{v})$, bidder j's probability of winning the auction with the bid $\beta^j(\tilde{v})$ is equal to zero. That is, $\prod_{k\in N\setminus\{j\}} p^k(\beta^j(\tilde{v})) = 0$. In addition, bidder i has a nonzero probability of winning with the bids $\beta^i(v)$ for all $v > \underline{v}$. That is, $\prod_{k\in N\setminus\{i\}} p^k(\beta^i(v)) > 0$ for all $v > \underline{v}$. If β^i is an equilibrium, then it is necessary that $\pi^i(\beta^i(v),v) \geq 0$ for all $v \in \mathscr{V}$ since there is always a bid that wins with zero probability. Notice that for $\hat{v} \in (\underline{v},\tilde{v})$,

$$\pi^i(\beta^i(\tilde{v}),\tilde{v}) \geq \pi^i(\beta^i(\hat{v}),\tilde{v}) > \pi^i(\beta^i(\hat{v}),\hat{v}) \geq 0.$$

The first weak inequality follows since if β^i is an equilibrium, then it must be at least as good for bidder i with value $\tilde{v}$ to bid $\beta^i(\tilde{v})$ as any other possible bid. The inequality in the second line follows because the function $\pi^i(b,\cdot)$ is increasing in its second argument for all b with a nonzero probability of winning. These inequalities establish that $\tilde{v} > \beta^i(\tilde{v})$.

It is straightforward to see that bidder j has a nonzero probability of winning with the bid $\beta^i(\tilde{v})$. Thus, $\pi^j(\beta^i(\tilde{v}),\tilde{v}) > 0$ since $\tilde{v} > \beta^i(\tilde{v})$ and j's probability of winning with the bid $\beta^i(\tilde{v})$ is nonzero. However, this violates the assumption that β^j is an equilibrium bidding strategy since $\pi^j(\beta^i(\tilde{v}),\tilde{v}) > \pi^j(\beta^j(\tilde{v}),\tilde{v}) = 0$.

Now I show that this common bid for $\underline{v}$ must be equal to $\underline{v}$. Let $\underline{b} = \beta^i(\underline{v})$ for all $i \in N$. Suppose that $\underline{b} > \underline{v}$. Then by continuity, there exists a $v > \underline{v}$ and an $i \in N$ such that $\beta^i(v) > v$ and i's probability of winning with the bid $\beta^i(v)$ is nonzero. However, β^i could not be an equilibrium since bidder i's payoff is negative for the value v.

Now suppose that $\underline{b} < \underline{v}$. Since the probability of winning with the bid $\underline{b}$ is zero, the payoff to a bidder in equilibrium with value $\underline{v}$ is zero. By the continuity implied by differentiability, there exists a bid b and an $i \in N$ such that $\underline{v} > b > \beta^i(\underline{v})$ and i's probability of winning with the bid b is nonzero. Thus, β^i is not an equilibrium since a bidder with value $\underline{v}$ is better off bidding b than $\beta^i(v)$. Hence, part (i) is shown since both $\underline{b}>\underline{v}$ and $\underline{b}<\underline{v}$ are inconsistent with equilibrium.

To prove part (ii), I must show that if $(\beta^1,\ldots,\beta^n)$ is an equilibrium strategy profile, then there are no $i,j\in N$ such that $\beta^i(\bar{v})>\beta^j(\bar{v})$. Let bidder i be a bidder who submits the highest bid. That is, $i=\arg\max_{k\in N}\{\beta^k(\bar{v})\}$. Suppose that there exists a $j\in N$ such that $\beta^i(\bar{v})>\beta^j(\bar{v})$. Then

$$[\bar{v}-\beta^i(\bar{v})]\frac{\prod_{k\in N}p^k(\beta^i(\bar{v}))}{p^j(\beta^i(\bar{v}))}=[\bar{v}-\beta^i(\bar{v})]\frac{\prod_{k\in N}p^k(\beta^i(\bar{v}))}{p^i(\beta^i(\bar{v}))}, \tag{A.1}$$

$$\geq[\bar{v}-\beta^j(\bar{v})]\frac{\prod_{k\in N}p^k(\beta^j(\bar{v}))}{p^i(\beta^j(\bar{v}))}, \tag{A.2}$$

$$>[\bar{v}-\beta^j(\bar{v})]\frac{\prod_{k\in N}p^k(\beta^j(\bar{v}))}{p^j(\beta^j(\bar{v}))}. \tag{A.3}$$

The equality in Eq. (A.1) follows since $p^i(\beta^i(\bar{v}))=p^j(\beta^i(\bar{v}))=1$. The weak inequality in Eq. (A.2) follows because if β^i is an equilibrium then $\pi^i(\beta^i(\bar{v}),\bar{v})\geq\pi^i(\beta^j(\bar{v}),\bar{v})$. The inequality in Eq. (A.3) follows because $p^i(\beta^j(\bar{v}))<p^j(\beta^j(\bar{v}))=1$ since $\beta^i(\bar{v})>\beta^j(\bar{v})$. Notice that these expressions imply that $\pi^j(\beta^i(\bar{v}),\bar{v})>\pi^j(\beta^j(\bar{v}),\bar{v})$. However, this is not consistent with an equilibrium profile.

Part (iii): Since the bidding strategies are increasing and differentiable, the inverses of the bidding strategies $z^1,\ldots,z^n$ are increasing and differentiable. Thus, for all $i\in N$, π^i is differentiable in its first argument. Hence (iii) must hold if β is an increasing and differentiable equilibrium. □

Proof of Proposition 1. *Proof of Part (i)*: I use the following Lemma in proving part (i).

Lemma 2. *If there exists $v\in(\underline{v},\bar{v})$ such that $\beta^i(v)\leq\beta^j(v)$, then there exists $\tilde{v}\in[v,\bar{v}]$ such that $\beta^i(\tilde{v})=\beta^j(\tilde{v})$ and $\beta^{i\prime}(\tilde{v})\geq\beta^{j\prime}(\tilde{v})$.*

Proof. Case I: First consider the case where $\beta^i(v)<\beta^j(v)$. Let $\tilde{v}=\min\{\xi\in[v,\bar{v}]:\beta^i(\xi)=\beta^j(\xi)\}$. Such a $\tilde{v}$ exists since $[v,\bar{v}]$ is a compact set and part (ii) of Lemma 1 implies that $\beta^i(\bar{v})=\beta^j(\bar{v})$. By the definition of $\tilde{v}$ and continuity, $\beta^i(\hat{v})<\beta^j(\hat{v})$, for all $\hat{v}\in[v,\tilde{v})$. Thus,

$$\frac{\beta^i(\tilde{v})-\beta^i(\hat{v})}{\tilde{v}-\hat{v}}>\frac{\beta^j(\tilde{v})-\beta^j(\hat{v})}{\tilde{v}-\hat{v}},\quad\forall\hat{v}\in[v,\tilde{v}),$$

which implies $\beta^{i\prime}(\tilde{v})\geq\beta^{j\prime}(\tilde{v})$.

Case II: Now consider the case where $\beta^i(v)=\beta^j(v)$. If $\beta^{i\prime}(v)\geq\beta^{j\prime}(v)$, then the proof is complete. If $\beta^{i\prime}(v) < \beta^{j\prime}(v)$, then there exists $v^* \in (v,\bar{v})$ such that $\beta^i(v^*) < \beta^j(v^*)$ by the continuity of the bidding functions. The proof is completed by applying the arguments in case I to v^* in place of v. □

By Lemma 2, the proof is complete if for all $v \in (\underline{v},\bar{v}]$ such that $\beta^i(v) = \beta^j(v)$ it cannot be the case that $\beta^{i\prime}(v) \geq \beta^{j\prime}(v)$.

Part (iii) of Lemma 1 states that if $(\beta^1,\ldots,\beta^n)$ is an increasing and differentiable equilibrium profile, then $\pi^i_1(\beta^i(v),v) = 0$. Notice that $\pi^i_1(\beta^i(v),v) = 0$ can be written as

$$[v - \beta^i(v)]\left(\sum_{k\in N} \frac{p^{k\prime}(\beta^i(v))}{p^k(\beta^i(v))} - \frac{p^{i\prime}(\beta^i(v))}{p^i(\beta^i(v))}\right) - 1 = 0, \quad \forall v \in (\underline{v},\bar{v}].$$

Thus, for v such that $\beta^i(v) = \beta^j(v) = b$, it must be that $p^{i\prime}(b)/p^i(b) = p^{j\prime}(b)/p^j(b)$. This can be written as

$$\frac{f^i(v)}{F^i(v)\beta^{i\prime}(v)} = \frac{f^j(v)}{F^j(v)\beta^{j\prime}(v)}.$$

Thus, $\beta^{i\prime}(v) < \beta^{j\prime}(v)$ since

$$\frac{f^i(v)}{F^i(v)} < \frac{f^j(v)}{F^j(v)}$$

by $SMTP_2$.

Proof of Part (ii). Bidder i's probability of winning the auction is

$$\int_{\underline{v}}^{\bar{v}} \prod_{h\in N\setminus\{i\}} p^h(\beta^i(v)) f^i(v)\, dv.$$

Through a transformation of variables, this probability can be written as

$$\int_{\underline{b}}^{\bar{b}} p^{i\prime}(b) \prod_{h\in N\setminus\{i\}} p^h(b) db,$$

where $\beta^i(\underline{v}) = \underline{b}$ and $\beta^i(\bar{v}) = \bar{b}$. Notice that $\pi^i_1(\beta^i(v),v) = 0$ can be written as

$$[z^i(b) - b]\sum_{k\in N} p^{k\prime}(b) \prod_{h\in N\setminus\{k\}} p^h(b) - p^{i\prime}(b) \prod_{h\in N\setminus\{i\}} p^h(b) - 1 = 0. \tag{A.4}$$

Since $[z^j(b) - b] > [z^i(b) - b]$ for all $b \in (\underline{b},\bar{b})$ by part (i), Eq. (A.4) implies that

$$p^{j\prime}(b) \prod_{h\in N\setminus\{j\}} p^h(b) > p^{i\prime}(b) \prod_{h\in N\setminus\{i\}} p^h(b) \tag{A.5}$$

for all $b \in (\underline{b},\bar{b})$. Thus,

$$\int_{\underline{b}}^{\bar{b}} p^{j\prime}(b) \prod_{h \in N\setminus\{j\}} p^h(b)\mathrm{d}b > \int_{\underline{b}}^{\bar{b}} p^{i\prime}(b) \prod_{h \in N\setminus\{i\}} p^h(b)\,\mathrm{d}b.$$

Hence, part (ii) is shown.

Proof of Part (iii). Notice that by dividing both sides of Eq. (A.5) by $\prod_{h \in N\setminus\{i,j\}} p^h(b)$ yields

$$\frac{p^{j\prime}(b)}{p^j(b)} > \frac{p^{i\prime}(b)}{p^i(b)}$$

for all $b \in (\underline{b},\bar{b})$. This is equivalent to $\mathrm{d}\log p^j(b)/\mathrm{d}b > \mathrm{d}\log p^i(b)/\mathrm{d}b$, for all $b \in (\underline{b},\bar{b})$. Since $\log p^j(\bar{b}) = \log p^i(\bar{b})$, it must be that $\log p^j(b) < \log p^i(b)$ for all $b \in (\underline{b},\bar{b})$. Therefore, $p^j(\bar{b}) < p^i(\bar{b})$ for all $b \in (\underline{b},\bar{b})$.

Proof of part (iv). The expected payoff of bidder i is

$$\int_{\underline{v}}^{\bar{v}} [v - \beta^i(v)] \prod_{h \in N\setminus\{i\}} p^h(\beta^i(v)) f^i(v)\,\mathrm{d}v.$$

Through a transformation of variables, the expected payoff can be written as

$$\int_{\underline{b}}^{\bar{b}} [z^i(b) - b] p^{i\prime}(b) \prod_{h \in N\setminus\{i\}} p^h(b)\,\mathrm{d}b.$$

By part (i) it is known that $[z^j(b) - b] > [z^i(b) - b]$, for all $b \in (\underline{b},\bar{b})$. This in addition to Eq. (A.5) above complete the proof.

It remains to be shown that parts (ii)–(iv) hold for second-price or open auctions. To see that part (ii) holds notice that bidder i's probability of winning the auction can be written as

$$E\Big[\prod_{k \in N\setminus\{i\}} p^k(\beta^i(V_i)) | t_i\Big].$$

This expectation is increasing in the value of t_i by Eq. (2). In second-price auctions the equilibrium strategy is $\beta^k(v) = v$ for all $k \in N$ and all values v. Thus, $p^k(\beta^k(v)) = F^k(v)$ for all $k \in N$ and all values v. Part (iii) follows for second-price auctions since for $t_i < t_j$, $p^i(\beta^i(v)) = F^i(v) > F^j(v) = p^j(\beta^j(v))$ for all v. Part (iv) follows from the fact that the expected payoff of bidder i can be written as

$$E\left[\int_{\underline{v}}^{V_i}[V_i-b]\mathrm{d}H(b)|t_i\right],$$

where $H(b)=\prod_{k\in N\setminus\{i\}}p^k(b)$. Notice that this expectation is increasing in t_i by Eq. (2). □

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