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A Model of Auction Contracts with Liquidated Damages*

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This paper develops an auction model in which the winning bidder has an opportunity to cancel the transaction and pay damages to the seller. In the event of a default on the auction contract, the winning bidder pays liquidated damages or loses a posted deposit. When renegotiation is possible, increasing the deposit has no effect on the seller's payoff unless the seller has some bargaining power and exogenously receives some information about the winning bidder. Under these conditions the seller's payoff is decreasing in the level of the deposit. *Journal of Economic Literature* Classification Number: D44.

1. INTRODUCTION

Auctions often require a bond or deposit to be posted before a bidder is certified. Presumably these terms are meant to protect the seller of the object at auction from default on a winning bid. An example of such a default occurred in the Federal Communications Commission sale of radio spectrum rights during the summer of 1994. A number of bidders were unwilling or unable to make a down payment two weeks after the spectrum auctions and were declared in default. The model presented in this paper demonstrates that the seller cannot increase her payoff by increasing the level of the deposit.

The submission of a bid in an auction marks the beginning of a contract between a bidder and a seller. The contract specifies the rules for selecting the winning bidder and for setting the price that the winning bidder must pay for the good. After the submission of bids, the contract binds both the winning bidder and the seller to complete the transaction at the specified price. In many auctions, deposits must be posted as security against the winning bidder's failure to complete his contractual obligations. Similarly,

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the auction contract can specify liquidated damages. The legal term "liquidated damages" describes damages for default that are specifically defined by a contract. Both liquidated damages and deposits are ex ante specifications of the seller's remedy in the event that the bidder breaches the auction contract.

Deposits and liquidated damages not only serve as penalties for failing to complete the terms of the auction contract, but they also define the maximum loss that can be imposed on a bidder. If the bidder expects a loss greater than the liquidated damages or deposit, then he will default on the contract. Thus, a model of liquidated damages or deposits can also be interpreted as a model of limited liability; that is, the model specifies a maximum loss that can be sustained by the winning bidder. Limited liability may result from a bidder's wealth constraint. In the model of this paper I use the term "deposit" rather than the synonymous terms "liquidated damage" and "limited liability."

In the main result of the paper, I analyze how changes in the deposit affect the seller's expected revenue from the auction. A large literature has developed around the comparison of the expected revenue generated by different auction rules. For a review of the auction literature see McAfee and McMillan [11] and Milgrom [13]. Milgrom and Weber [14] compare different auctions when bidders' valuations are affiliated. Harstad [3] conducts a similar comparison, when the number of bidders is endogenous. Matthews [10] provides this comparison when bidders have differing degrees of risk aversion. Riley [15] shows that the seller can extract more revenues if she makes the bidders' payment contingent on private information that is revealed after the bidding.

Spulber [16] studies the enforcement of auction contracts. In his model, the bidders' values differ only in the size of a cost overrun that occurs with common probability. He finds that when contract enforcement is weak (damages for default are low), there are no increasing symmetric equilibrium bidding strategies. Harstad and Rothkopf [4] analyze a common value auction where the winning bid can be withdrawn if the winner regrets his bid after observing the bids of his rivals. They show that the seller receives higher expected revenue when withdrawal is possible.

I find that increasing the deposit has either no effect or a negative effect on the seller's expected revenue from the auction. Raising the deposit will negatively impact revenues if (1) the winning bidder's private information is exogenously revealed to the seller after the auction but before renegotiation, and (2) the seller has the bargaining power in the renegotiation stage to "link" the renegotiated price to the winning bidder's private information. By the linkage principle, a strengthening of the "linkage" between a bidder's private information and his expected payment causes an increase in the seller's revenue. Therefore, decreasing the deposit increases the seller's payoff, since a decrease in the deposit increases the probability of renegotiation and, hence, the "linkage" between the bidder's expected payment and his private information.

2. The Model

The seller wishes to award an indivisible object to one of a group of bidders. However, the seller does not know the value of the contract to these bidders. The object could be a contract for the provision of goods or services and is awarded through a sealed-bid or open auction. After the winning bidder is selected but before the execution of the contract, new information becomes publicly available that affects the valuations of the bidders. The new public information is revealed to both the bidders and the seller. Between the revelation of the new information and the completion of the auction transaction, the winning bidder has an opportunity to default at the cost of the posted deposit. While I only consider the case where the buyer can default, there are circumstances where the seller may also wish to default and renegotiate the auction contract. Because of the additional complexity introduced if the seller were allowed to default, I will assume in this paper that the auction contract requires "specific performance" on the part of the seller. "Specific performance" is a legal term describing the most strict level of contract enforcement where parties are forced to carry out the terms of the contract. If the bidder defaults on the contract, then the seller keeps the deposit and might have the opportunity to renegotiate with any of the original bidders (including the bidder who canceled the contract). However, I assume that the auctioneer cannot compel nonwinning bidders to resubmit their original bids.

The set $N = \{1, ..., n\}$, $n \geqslant 2$, indexes the bidders.[1] I assume that bidders are risk-neutral. V_i denotes the value of the object at auction to bidder $i \in N$, while V_0 denotes the value to the seller. The realizations of real-valued random variables X_i and S determine V_i through the function v:

$$V_i = v(S, X_i), \qquad \forall i \in N.$$

The realization of S determines V_0 through the function v_0:

$$V_0 = v_0(S).$$

The distribution of X_i has support equal to $\tilde{X} = [\underline{x}, \bar{x}]$, and the distribution of S has support equal to $\mathbf{R}$. The random variable X_i denotes bidder i's private information, and the random variable S is information that is

[1] The notation follows that of Milgrom and Weber [14] wherever possible.

publicly revealed after the bidding but before the completion of the transaction. Note that the function v is common to all bidders. If the revelation of S is good news, then the values of all of the bidders increases. This symmetry assumption rules out the revelation of information specific only to the winning bidder's value.

ASSUMPTION 1. *The functions v and v_0 are differentiable and increasing in their arguments.*

ASSUMPTION 2. *$X_1, ..., X_n$ and S are independent random variables with positive density functions. Furthermore, $X_1, ..., X_n$ are identically distributed.*

Assumptions 1 and 2 make the model similar to the private values model, in that each bidder's value is independent of other bidders' private information, and the private information variables X_i are independent of S. I have not been able to show the existence of an equilibrium bidding strategy without Assumption 2.

ASSUMPTION 3. *The functions v and v_0 have the property that $\forall \hat{v} \in \mathbf{R}$, $\forall x \in \tilde{X}$, $\exists s \in \mathbf{R}$ such that $v(s, x) = \hat{v}$ and $\exists s' \in \mathbf{R}$ such that $v_0(s') = \hat{v}$.*

Assumption 3 implies that the supports of the buyer's and the seller's valuations are both equal to the set of real numbers. The assumption helps ensure that the probability of default is strictly less than one.

ASSUMPTION 4. *$E[V_i \mid X_i = x] \geqslant 0$, $\forall i \in N$ and $\forall x \in \tilde{X}$.*

ASSUMPTION 5. *The expected gains from completing the transaction are nonnegative and bounded from above; that is, there exists an $m \geqslant 0$ such that*

$$m \geqslant v(s, x) - v_0(s) \geqslant 0, \qquad \forall x \in \tilde{X}, \qquad \forall s \in \mathbf{R}.$$

The timing of the game is as follows. Each bidder $i \in N$ observes his private information X_i and submits a bid. Thus, a bidder's strategy can be expressed as a function of his private information. The bids are revealed and the winning bidder and the auction price are determined. The realization of S is publicly revealed after the auction. The winning bidder then decides whether or not to proceed with the transaction at the auction price. If he defaults, then the winning bidder loses the deposit, denoted δ, to the seller.

The public revelation of S suggests that it might be possible to construct bids that are contingent on the realization of S so as to avoid defaults on the auction contract. That is, each bidder would submit a payment schedule indicating the price to be paid at each realization of S. Such a contract could be ruled out by a number of assumptions used in the incomplete contracts literature. For instance, writing or enforcing a contract that makes the price contingent on S may be too costly.[2]

The analysis focuses on symmetric subgame perfect equilibrium bidding strategies that are increasing in the private information of a bidder.[3] Thus, without loss of generality, attention is restricted to the strategy of bidder 1. Define $Y_1, ..., Y_{n-1}$ as a reordering of $X_2, ..., X_n$ such that $Y_1 \geqslant \cdots \geqslant Y_{n-1}$ so that Y_1 denotes the highest private information value for bidders $N\backslash\{1\}$. Y_1 is an order statistic for the random variables $X_2, ..., X_n$, with density $g(y) = (n-1)F_X(y)^{n-2}f_X(y)$ and distribution function $G(y) = F_X(y)^{n-1}$, where f_X and F_X denote the common density function and distribution function of $X_1, ..., X_n$.

In general it is not possible to derive closed form solutions for the equilibrium bidding strategies. However, in the following example such a solution is possible.

EXAMPLE 1. Let $v(x, s) = s + x$ and $v_0(s) = s$. The bidders commonly believe that S has a logistic distribution so that $F(s) = 1/(1 + e^{-s})$. Furthermore, let $\delta > 0$ denote the deposit. For the purposes of this example, I assume that once default occurs the seller has no other opportunities to sell the object. (Later in the paper this is defined as the no renegotiation (NR) case.) If q is the price determined by the auction, then after observing s, the winning bidder must decide between completing the contract and receiving a payoff of $x + s - q$ or defaulting and receiving a payoff of $-\delta$. Hence, the winning bidder would choose to default if and only if the realization of s is less that $q - \delta - x$. Let $\Lambda(q, x)$ denote a bidder's expected payoff conditional on the event of winning with the bid q and that his private information and the highest value of the other bidders' private information are equal to x. In a second-price or open ascending value auction, it is a dominant strategy to bid so that $\Lambda(q, x) = 0$. Therefore, the bidding strategy $\beta(\cdot)$ yields a symmetric equilibrium bidding strategy for this example if

$$\Lambda(\beta(x), x) = \int_{\beta(x)-\delta-x}^{\infty} [x + s - \beta(x)] \frac{e^{-s}}{(1+e^{-s})^2} \, ds - \frac{\delta}{1 + e^{-(\beta(x)-\delta-x)}} = 0$$

for all x. Solving for $\beta(x)$ in the expression above gives $\beta(x) = x + \delta - \ln(e^{\delta} - 1)$. The equilibrium bid is decreasing in δ. Notice that $\lim_{\delta \to 0} \beta(x) = \infty$ and $\lim_{\delta \to \infty} \beta(x) = x$.

[2] See Hart and Holmstrom [5] for a discussion of why contracts may be incomplete.

[3] It has been shown by Milgrom [12] and Bikhchandani and Riley [1] that in some situations asymmetric equilibria exist. However, to simplify the analysis I restrict attention to symmetric equilibria.

3. Defaulting and Renegotiation

In the event of a default by the winning bidder, I consider four possibilities. The first possibility is that there is no renegotiation (NR) between the seller and any of the bidders. Hence, the object at auction would revert back to the seller in the event of a default. In equilibrium, if the object does revert back to the seller, then the outcome is inefficient, since according to Assumption 5 the seller's value for the object is less than any of the bidders' values (except possibly a bidder with private information $\underline{x}$).

The other possibilities I consider involve renegotiation between the seller and the winning bidder. In equilibrium the winning bidder has the highest value for the object at auction before and after S is revealed. Let the function $p(s, x, y)$ denote the renegotiated price from the winning bidder's perspective when s is the realization of S, x is the private information of the winning bidder, and y is the highest of the private information values of the nonwinning bidders; that is, if bidder i is the winning bidder, then $y = \max\{x_j\}_{j \neq i}$. The construction of p that follows assumes that renegotiation occurs after default and hence after the deposit is paid to the seller. As I discuss below, the results of the model will be unaffected by alternatively assuming that renegotiation occurs prior to default. The three possible outcomes of renegotiation that I consider are:

Strong Buyer (SB): $p(s, x, y) = v_0(s)$.

Strong Seller (SS): $p(s, x, y) = v(s, x)$.

Competitive Buyers (CB):

$$p(s, x, y) = \begin{cases} v(x, y), & \forall x \geqslant y \\ v(s, x), & \text{otherwise.} \end{cases}$$

Under (SB) the buyer extracts all of the surplus in the renegotiation stage. Under (SS) the seller extracts all of the surplus in the renegotiation stage. For the seller to extract all of the surplus in renegotiation she must be able to observe the winning bidder's private information rather than simply infer it from the winning bid. Hence, under (SS) it must be the case that the pretransaction relationship between the winning bidder and the seller provides an opportunity for the seller to observe x.

The (CB) case is somewhat more complicated than the others. The intuition underlying the (CB) case arises from a situation where the seller induces the winning bidder to pay the maximum that any other bidder is willing to pay (as long as it does not exceed the winning bidder's valuation) by threatening to hold a second-stage auction. A more complete description of this intuition is presented in the Appendix. Under (CB), the object is sold to the winning bidder at the maximum any other bidder is willing to pay, if $x \geqslant y$. If $x < y$, then the object is not sold to the winning bidder. In terms of the bidder's payoff function, (CB) is equivalent to the case where if $x < y$, then the winning bidder gets the object and pays the price $v(s, x)$. To see this, note that when the winning bidder does not receive the object through renegotiation his payoff is $-\delta$, where δ is the level of the deposit. If the winning bidder does receive the object at the price $v(s, x)$, then his payoff is $v(s, x) - v(s, x) - \delta(= -\delta)$. Therefore, to simplify the modelling of the bidder's payoff function under (CB) I define $p(s, x, y) = v(s, x)$ whenever $x < y$ since it gives rise to the payoff function as described for the (CB) case.

Now suppose that renegotiation occurs prior to default and the resulting price is $\hat{p}(s, x, y)$. Under (SB), $\hat{p}$ is equal to the lowest acceptable price to the seller. Hence, $\hat{p}(s, x, y) = v_0(s) + \delta$, since the seller knows that if no price is agreed to, then her payoff is her value for the object plus the deposit. If renegotiation occurs prior to default the buyer's payoff is $v(s, x) - v_0(s) - \delta$, and the seller's revenue would be $v_0(s) + \delta$. These are exactly the same payoffs that arise under (SB) when renegotiation occurs *after* default. Therefore, under (SB) both the buyer's payoff and the seller's revenue are unaffected by the timing of renegotiations. Similar arguments can be made for the (SS) and (CB) cases.

Notice that under (SS),(SB), and (CB) the function p has the following properties:

(P1) p is increasing in s, nondecreasing in x and y, and differentiable in all its variables except possibly where $x = y$,

(P2) the difference $v(s, x) - p(s, x, y)$ is nondecreasing in x and nonincreasing in y.

Let q denote the price determined by the initial auction, and let δ denote the deposit. Faced with the decision of whether or not to default under (SS),(SB), and (CB), the winning bidder will compare his expected payoff if he completes the terms of the auction contract (i.e., $v(s, x) - q$) with his payoff if he defaults (i.e., $v(s, x) - p(s, x, y) - \delta$). Hence, the bidder should default if $q > p(s, x, y) + \delta$. I define the function s^*: $\mathbf{R} \times \tilde{X} \times \tilde{X} \to \mathbf{R}$ such that $s^*(q - \delta, x, y)$ is the unique s that solves the equation

$$q = p(s, x, y) + \delta. \tag{1}$$

When s^* is the realization of S, the winning bidder is indifferent between completing the transaction at the auction price and defaulting.

Assumption 3 and property (P1) imply that s^* is well defined in Eq. (1). Furthermore, s^* is continuous and the implicit function theorem implies that s^* is differentiable at $(q - \delta, x, y)$ if $x \neq y$. Let $h(x) = s^*(q - \delta, x, x)$. Note that $h'(x)$ is well defined. This result is immediate in the cases of (SS) and (SB). To see that $h'(x)$ is well defined under (CB) define a function $\tilde{p}$ as

$$\tilde{p}(s, x) = p(s, x, x) = v(s, x).$$

Under (CB), $s^*(q-b, x, x)$ satisfies $q = \tilde{p}(s^*(q-\delta, x, x), x) + \delta$. By the implicit function theorem $h'(x)$ is well defined.

If the winning bidder observes a realization of S greater than or equal to $s^*(q-\delta, x, y)$, he will complete the transaction at the auction price; if he observes a realization of S less than $s^*(q-\delta, x, y)$, he will announce his intention to default on the contract.

LEMMA 1. *The equilibrium bidding strategy under* (NR) *is the same as the equilibrium bidding strategy under* (SS).

Proof. Under both (SS) and (NR) the winning bidder's payoff if he completes the terms of the auction contract is $v(s, x) - q$. When the winning bidder defaults on the auction contract, his payoff under (SS) is $v(s, x) - p(s, x, y) - \delta = -\delta$ while his payoff under (NR) is $-\delta$. Therefore, any bidding strategy that maximizes a bidder's payoff under (SS) also maximizes his payoff under (NR). ∎

In order to complete the definition of the (NR) case it is necessary to define the functions p and s^* for that case. Lemma 1 is the motivation for the definition of p for the (NR) case that follows.

No Renegotiation (NR): $p(s, x, y) = v(s, x)$

Hence, in terms of a bidder's payoff and strategy the (SS) and (NR) are the same. In particular s^* has the same definition in both cases. While (SS) and (NR) are equivalent in terms of a bidder's expected payoff, (SS) and (NR) are very different from the seller's perspective. Under (SS) the seller receives $v(s, x) + \delta - v_0(s)$ in the event of a default; under (NR) the seller only receives δ.

4. Characterization of Symmetric Equilibrium Bidding Strategies

4.1. *Second-Price Auctions*

In a second-price auction, the highest bidder wins and pays an amount equal to the second highest bid. In an open auction, the active bid is orally raised until only one bidder remains active. A bidding strategy in an open auction is the bid at which a bidder would drop out of the auction. If a bidder's expected payoff somehow depends on the private information of other bidders, then observing when other bidders drop out of the auction would cause a bidder to update his estimate of his own expected payoff. However, the definition of v together with Assumption 2 imply that second-price auctions and open ascending value auctions are strategically equivalent as long as the renegotiated price is independent of the private information of the other bidders. That is, open and second-price auctions are equivalent under (NR),(SS), and (SB).

A bidder's strategy is a function of his private information. Suppose that β is an increasing function that represents the symmetric equilibrium strategy. If x is the realization of X_1 and y is the realization of Y_1, then in equilibrium bidder 1 would win if $x > y$ and would pay $\beta(y)$. The following analysis demonstrates the existence of a symmetric equilibrium strategy β that is increasing in the bidder's private information.

Suppose bidder 1 is the highest bidder and is required to pay price q. If $X_1 = x$, $Y_1 = y$ and $S = s$, then he accepts the object when $s \geqslant s^*(q-\delta, x, y)$ and defaults when $s < s^*(q-\delta, x, y)$. Thus, if bidder 1 wins the auction and the other bidders use the same increasing strategy β, then his expected payoff conditional on $X_1 = x$ and $Y_1 = y$, denoted $\tilde{\pi}$, is written

$$\tilde{\pi}(q, x, y) = \int_{s^*(q-\delta, x, y)}^{\infty} [v(s, x) - q]\, f(s)\, ds + \int_{-\infty}^{s^*(q-\delta, x, y)} [v(s, x) - p(s, x, y) - \delta]\, f(s)\, ds,$$

where f enotes the density function of S.

In second-price auctions, the price is equal to the highest of the losing bids, $\beta(y)$. Let $\pi^{SP}(z, x)$ denote the expected payoff of a bidder with private information x but who bids as if his private information were equal to z; that is,

$$\pi^{SP}(z, x) = \int_{\underline{x}}^{z} \tilde{\pi}(\beta(y), x, y)\, g(y)\, dy.$$

(Note that I have suppressed the dependence of π^{SP} on β.) Recall that g denotes the density of Y_1. The following lemma demonstrates that sufficient conditions for a symmetric increasing equilibrium bidding strategy in a second-price auction can be expressed in terms of π^{SP}.

LEMMA 2. *Let $\pi^{SP}(z, x)$ be as above. If β is an increasing function satisfying*

(A1) $\pi_1^{SP}(z, x) \geqslant 0, \quad z < x$

(A2) $\pi_1^{SP}(z, x) \leqslant 0, \quad z > x$

(A3) $\tilde{\pi}(\beta(\underline{x}), \underline{x}, \underline{x}) = 0,$

then β is an equilibrium bidding strategy in the second-price auction. (The notation $\pi_i(z, x)$ represents the partial derivative of π with respect to its ith argument.)

(See the Appendix for the proof.)

The following proposition characterizes an equilibrium bidding strategy for a second-price auction.

PROPOSITION 1. *Suppose* (i) $\delta > 0$ *and* p *satisfies* (NR), (SS), *or* (CB), *or* (ii) $\delta > E[v(S, \bar{x}) - p(S, \bar{x}, \bar{x})]$ *and* p *satisfies* (SB). *For second-price auctions there exists a symmetric Nash equilibrium bidding strategy* β: $\tilde{X} \to \mathbf{R}$ *defined by the equation*

$$\int_{s^*(\beta(x)-\delta, x, x)}^{\infty} [v(s, x) - \beta(x)]\, f(s)\, ds + \int_{-\infty}^{s^*(\beta(x)-\delta, x, x)} [v(s, x) - p(s, x, x) - \delta]\, f(s)\, ds = 0. \quad (2)$$

(See the Appendix for the proof.)

4.2. *First-Price Auctions*

In a first-price auction, the highest bidder wins and is required to pay an amount equal to his own bid. Let $\pi^{\mathrm{FP}}(z, x)$ denote the expected payoff of a bidder with private information x who bids as if his private information were equal to z; that is,

$$\pi^{\mathrm{FP}}(z, x) = \int_{\underline{x}}^{z} \tilde{\pi}(\beta(z), x, y)\, g(y)\, dy.$$

(Note that I have suppressed the dependence of π^{FP} on β.)

The following lemma gives the conditions required for the existence of an equilibrium. The proof proceeds along the same lines as the proof of Lemma 2.

LEMMA 3. *Let* $\pi^{\mathrm{FP}}(z, x)$ *be as above. If* β *is an increasing function satisfying*

(B1) $\pi_1^{\mathrm{FP}}(z, x) \geqslant 0, \quad z < x$

(B2) $\pi_1^{\mathrm{FP}}(z, x) \leqslant 0, \quad z > x$

(B3) $\tilde{\pi}(\beta(\underline{x}), \underline{x}, \underline{x}) = 0,$

then β *is an equilibrium bidding strategy in the first-price auction.*

The analysis of the first-price auction is considerably more complicated than the second-price auction. In particular, I have not been able to show that the conditions of Lemma 3 are satisfied for first-price auctions using the conditions imposed up to now. Thus, to make the first-price problem more tractable I make the following assumptions.

[*Monotone Hazard Rate* (MH)] $f(s)/[1-F(s)]$ is nondecreasing in s, and $\int_s^\infty [1-F(\xi)]\, d\xi < \infty$, for all s.[4]

[*Separability* (S)] $v(s,x) = \psi(x) + \phi(s)$, where $\psi'(x) > 0$, $\phi'(s) > 0$, $\phi''(s) \leqslant 0$, and $\lim_{s \to \infty} \phi(s)[1-F(s)] = 0$.

The monotone increasing hazard rate assumption (MH) is a regularity condition that is common in models where incentive compatibility constraints are imposed.

PROPOSITION 2. *Let p satisfy* (NR), (SS), (SB), *or* (CB). *If* (S) *and* (MH) *hold and* β *is a solution to the differential equation*

$$\beta'(x) = \left\{ \frac{\int_{s^*(\beta(x)-\delta,x,x)}^{\infty} [v(s,x) - \beta(x)]\, f(s)\, ds}{\int_{\underline{x}}^{x} \int_{s^*(\beta(x)-\delta,x,x)}^{\infty} f(s)\, g(y)\, ds\, dy} + \frac{\int_{-\infty}^{s^*(\beta(x)-\delta,x,x)} [v(s,x) - p(s,x,y) - \delta f(s)]\, ds}{\int_{\underline{x}}^{x} \int_{s^*(\beta(x)-\delta,x,x)}^{\infty} f(s)\, g(y)\, ds\, dy} \right\} g(x), \qquad (3)$$

with initial condition

$$\int_{s^*(\beta(x)-\delta,\underline{x},x)}^{\infty} [v(s,\underline{x}) - \beta(\underline{x})]\, f(s)\, ds + \int_{-\infty}^{s^*(\beta(x)-\delta,\underline{x},x)} [v(s,\underline{x}) - p(s,\underline{x},\underline{x}) - \delta]\, f(s)\, ds = 0,$$

then β *is a symmetric equilibrium bidding strategy for a first-price auction.*

(See the Appendix for the proof.)

4.3. *Enforcement versus Nonenforcement of an Auction Contract*

The general assumption in this paper is that the auction contract is "partially enforced" in the sense that the winning bidder is not forced to pay the auction price after observing the realization of S. In the traditional auction

[4] This assumption is satisfied by uniform, exponential, and normal distributions. Assumption (MH) is slightly stronger than the standard assumption of a monotone nondecreasing hazard rate. The requirement that $\int_s^\infty [1-F(\xi)]\, d\xi$ converge is satisfied by any distribution where $f(s)/[1-F(s)]$ is nondecreasing and reaches a value greater than or equal to one.

framework where deposits do not appear, the auction contract is "perfectly enforced," in the sense that the winning bidder must pay the auction price and take the object. It seems plausible, therefore, that partially enforced auction contracts (described here) with large δ should be "similar to" auctions that are perfectly enforced. Indeed, under the mild regularity condition that $\lim_{\delta \to \infty} \delta F(s^*(\beta_\delta(x) - \delta, x, x) = 0$, the equilibrium bidding strategies characterized in Propositions 1 and 2 "converge" to the Milgrom and Weber [14] equilibrium bidding strategies.

Another limiting case arises when the auction contract is weakly enforced; that is, when δ is close to zero. As δ approaches zero, the equilibrium bid, if it exists, approaches infinity. Therefore, weak enforcement of the auction contract may lead to bids that are well above the expected value for the object at auction.

The following proposition demonstrates how changes in the level of the deposit affects the equilibrium bidding strategy and the probability of default.

PROPOSITION 3. *Let p satisfy* (NR), (SS), (SB), *or* (CB). *Consider two auctions (second-price or first-price) that are identical except for their level of the deposit δ and $\hat{\delta}$ with $\delta > \hat{\delta}$. Let β_δ and $\beta_{\hat{\delta}}$ be increasing, symmetric, and differentiable equilibrium bidding strategies. Then* $\beta_\delta(x) < \beta_{\hat{\delta}}(x)$, $s^*(\beta_\delta(x) - \delta, x, y) < s^*(\beta_{\hat{\delta}}(x) - \hat{\delta}, x, y)$, *and* $F(s^*(\beta_\delta(x) - \delta, x, y)) < F(s^*(\beta_{\hat{\delta}}(x) - \hat{\delta}, x, y))$, *for all* $x, y \in \tilde{X}$.

(See the Appendix for the proof.)

Proposition 3 asserts that equilibrium bid levels and the probability of default are both decreasing in the level of the deposit.

5. Revenue Comparisons

5.1. *Revenue Comparisons with Renegotiation*

Revenue comparisons have occupied a great deal of attention in the competitive bidding literature. The main building block of these results is the Linkage Principle. In the present context, revenue comparisons are made across auctions with different levels of the deposit. The Linkage Principle[5] exploits a statistical "linkage" between a bidder's private information and the informational basis of the price. An auction where the bidder with the highest value wins and pays a positive price, while the losing bidders pay nothing, is called a standard auction. The Linkage Principle applies to

[5] My exposition of the Linkage Principle has greatly benefited from Milgrom and Weber [14] and Milgrom [13].

all standard auctions. Thus, the application of the Linkage Principle requires the existence of an equilibrium bidding strategy β that is increasing in a bidder's private information.

Let $r(z, x)$ denote the expected benefit received from participating in the auction given that the other bidders use β and that bidder 1 receives private information x but bids $\beta(z)$; that is, $r(z, x) = E[V_1 | X_1 = x]G(z)$. (Recall that G is the distribution function of Y_1.) Let $w^A(z, x)$ denote the payment in the competitive bidding mechanism A for a bidder who receives private information x but bids $\beta(z)$. Thus, π^A (the expected payoff of a bidder participating in auction A) can be written in terms of r and w as $\pi^A(z, x) = r(z, x) - w^A(z, x)$.

LEMMA 4. (The Linkage Principle). *For two standard auctions* A *and* B, *suppose that for all* $x \in \tilde{X}$, $w^A(x, x) < w^B(x, x)$ *implies* $w_2^A(x, x) \geq w_2^B(x, x)$. *Then* $w^A(x, x) \geq w^B(x, x)$ *for all* $x \in (\underline{x}, \bar{x}]$, *and the expected payment by the winning bidder is larger in auction* A *than in* B.

(See Milgrom [13, Proposition 6].)

I use the Linkage Principle in the following proposition to relate the size of δ to the seller's expected revenue. The proposition implies that a seller does not increase her expected revenues by strengthening her enforcement of the auction contract.

PROPOSITION 4. *Consider an auction with a deposit in which the symmetric equilibrium strategy is increasing and differentiable in a bidder's private information*:

(a) *under* (SS), *the seller's payoff is a decreasing function of* δ

(b) *under* (CB) *and* (SB), *the seller's payoff is constant with respect to* δ.

Proof. Recall that $w^A(z, x)$ denotes the expected payment by a bidder with private information x who bids as if he had private information z in standard auction A. The proof is completed by applying the Linkage Principle.

Let $\delta > \hat{\delta}$ and suppose $w^A(z, x; \delta)$ and $w^A(z, x, \hat{\delta})$ denote the expected payments in two auctions that differ only in the deposit δ and $\hat{\delta}$, respectively. Let β_δ and $\beta_{\hat{\delta}}$ be increasing and symmetric equilibrium bidding strategies in these two auction, respectively. If $w(z, x)$ is defined as above, then in a first-price auction,

$$w^{FP}(z, x; \delta) = \int_{\underline{x}}^{z} \left\{ \beta_\delta(z)[1 - F(s^*(\beta_\delta(z) - \delta, x, y))] + \int_{-\infty}^{s^*(\beta_\delta(z) - \delta, x, y)} [p(s, x, y) + \delta]\, f(s)\, ds \right\} g(y)\, dy,$$

so that

$$w_2^{\mathrm{FP}}(x, x; \delta) = \int_{\underline{x}}^{x} \int_{-\infty}^{s^*(\beta_\delta(x) - \delta, x, y)} p_2(s, x, y)\, f(s)\, g(y)\, ds\, dy.$$

For a second-price auction,

$$w^{\mathrm{SP}}(z, x; \delta) = \int_{\underline{x}}^{z} \left\{ \beta_\delta(y)[1 - F(s^*(\beta_\delta(y) - \delta, x, y))] + \int_{-\infty}^{s^*(\beta_\delta(y) - \delta, x, y)} [p(s, x, y) + \delta]\, f(s)\, ds \right\} g(y)\, dy,$$

so that

$$w_2^{\mathrm{SP}}(x, x; \delta) = \int_{\underline{x}}^{x} \int_{-\infty}^{s^*(\beta_\delta(y) - \delta, x, y)} p_2(s, x, y)\, f(s)\, g(y)\, ds\, dy.$$

Then, by Proposition 3, $\delta > \hat{\delta}$ implies $s^*(\beta_\delta(x) - \delta, x, y) < s^*(\beta_{\hat{\delta}}(x) - \hat{\delta}, x, y)$ for all x, y. Hence, by the hypothesis of the linkage principle, the effect of δ depends on the sign of $p_2(s, x, y)$. Using the expression for $w_2^{\mathrm{FP}}(x, x)$ above; note that $w_2^{\mathrm{FP}}(x, x; \delta) < w_2^{\mathrm{FP}}(x, x; \hat{\delta})$ under (SS) since $p_2(s, x, y) > 0$. Therefore, the seller's revenue under (SS) is decreasing in δ. Under (CB) and (SB), $w_2^{\mathrm{FP}}(x, x; \delta) = w_2^{\mathrm{FP}}(x, x; \hat{\delta})$ since $p_2(s, x, y) = 0$. Therefore, the seller's revenue under (CB) and (SB) is constant with respect to δ. An identical argument applies to second-price auctions. ▮

From the proof of Proposition 4 it is clear that the effect of δ on the seller's payoff depends on the sign of $p_2(s, x, y)$ and not the level of $p(s, x, y)$ per se. Suppose that the renegotiated price is $p(s, x, y) = \alpha v(s, x) + (1 - \alpha)\, v_0(s)$. Assuming that the resulting equilibrium bidding strategy is increasing and differentiable, the seller's payoff would decrease in δ as long as $\alpha > 0$. Hence, the result depends not on the seller receiving a high price in renegotiation but rather on her ex post knowledge of the winning bidder's private information from an exogenous source and her ability to make the renegotiated price depend on that information.

Revenue equivalence fails under (SS) because the seller is assumed to be able to extract all the surplus in renegotiation whether or not the winning bidder submits a bid consistent with equilibrium. Hence, I assume that the seller's information regarding the winning bidder's private information comes from a source other than the winning bid. Under (SB) and (CB) revenue equivalence holds since the renegotiated price does not depend on the winning bidder's private information.

The result is similar to the main result of Riley [15] who shows that if ex post information on the buyer's private information can be related to the price paid by the buyer, then the seller can increase her revenue from the auction. Renegotiation provides an opportunity for the seller to link the price to the winning bidder's private information. When the renegotiated price p is positively related to knowledge of the winning bidder's private information from sources other than the winning bid, the "linkage" between the buyer's private information and the price he pays increases as the probability of renegotiation increases. The seller can increase the probability of renegotiation through a decrease in the level of the deposit.

Part (b) of Proposition 4 seems counterintuitive, especially under (SB). When the probability of default is "large" (close to one), a seemingly reasonable conjecture is that the seller's revenue is approximately the seller's payoff when default occurs. Under (SB) the seller's payoff in the event of a default is δ, the amount of the deposit. Hence, it seems reasonable to conclude that the seller's payoff should be increasing in δ when the probability of default is close to one. However, this argument fails because when the default probability is close to one the seller's payoff need not be approximately equal to δ. Even though the probability that the transaction is completed at the auction price is close to zero, the level of the bid is large. Hence, while the probability of completing the transaction at the auction price is small, the seller's payoff conditional on completion of the transaction has a nontrivial impact on the seller's overall expected payoff. Clearly this line of argument depends on the probability of default being less than one. If the winning bidder always defaults, then the seller's payoff under (NR), (SS), (SB), and (CB) is increasing in the level the deposit.

In many situations, bidders are required to submit a fraction γ their bid as a deposit rather than a fixed quantity such as δ. When the deposit is a fraction of the bid, the conclusions of Proposition 4 remain unchanged, whenever the equilibrium probability of default is a decreasing function γ. To see this, note that arguments presented in the proof of Proposition 4 are unaffected when δ is replaced by $\gamma\beta_\gamma(\cdot)$.

5.2. *Revenue Comparisons without Renegotiation*

By Lemma 1 it is clear that (NR) is equivalent to (SS) in terms of a bidder's payoff. This occurs because in the event of a default the winning bidder's payoff is the same under (NR) and (SS). The winning bidder is better off under both (CB) and (SB) than under either (SS) or (NR). From the seller's perspective (NR) and (SS) are quite different. In the event of a default the seller receives δ under (NR) and $\delta + v(s, x) - v_0(s)$ under (SS). Hence, by Assumption 5 the seller's payoff is greater under (SS) than under (NR).

In addition, the auction under (NR) is not a standard auction and the linkage principle cannot be applied since the bidder with the highest valuation for the object does not always take possession of the object. Consider Example 1 and assume that $X_1, \ldots, X_n$ are uniformly distributed on $[0,1]$. The payoff to the seller can be written as

$$\int_0^1 \int_0^x \Big\{\beta(y)[1 - F(\beta(y) - \delta - x)] + \delta F(\beta(y) - \delta - x)$$
$$- \int_{\beta(y) - \delta - x}^{\infty} s\, dF(s)\Big\}\, n(n-1)\, y^{n-2}\, dy\, dx.$$

Numerical integration of this expression indicates that in this example the seller's payoff is increasing in δ. Although I have not been able to prove a more general result, the following remark demonstrates that for δ close to zero the seller's gain under (NR) is close to zero.

Remark 1. In a first-price auction the seller's expected gain from the sale under (NR) when the winning bidder has private information value x can be written as

$$\int_{s^*(\beta(x) - \delta, x, y)}^{\infty} [v(s, x) - v_0(s)]\, f(s)\, ds$$
$$- \Big\{\int_{s^*(\beta(x) - \delta, x, y)}^{\infty} [v(s, x) - \beta(x)]\, f(s)\, ds - \delta F(x^*(\beta(x) - \delta, x, y))\Big\}.$$

In the expression above, the first integral term is the expected gain from trade. The term within the braces is the expected payoff to a winning bidder, which must be nonnegative. For δ close to zero, $\beta(x)$ is "large" and, thus, s^* must also be "large." However, when s^* is "large," the total expected gain from trade is close to zero since trade occurs with low probability. An identical argument applies to second-price auctions. Therefore, when there is no renegotiation (NR) the seller's payoff is "close" to zero when the deposit is "close" to zero. Hence, the seller would not set a deposit "close" to zero when there is no renegotiation following a default on the auction contract.

6. Conclusion

One of the implications of Proposition 4 is that when renegotiation follows a default on the auction contract, the seller's payoff is nonincreasing in the level of the deposit. Thus far I have assumed that defaulting and renegotiation are socially costless; that is, the process of defaulting and renegotiation do not require valuable resources. However, if the winning bidder's default and renegotiation costs (those costs other than δ) are positive and independent of his private information, then it is a straightforward application of Lemma 4 to show that under (CB) and (SB) the seller's payoff is increasing in δ. Lemma 4 implies that as long as the costs and payments associated with an auction are independent of the winning bidder's private information each bidder's expected payoff is unaffected by changes in δ. However, relatively lower values of δ result in relatively higher social costs associated with renegotiation since the probability of a default is decreasing in δ. Since each bidder's expected payoff is independent of δ, the larger social costs associated with lower deposits must come out of the seller's share of the expected gains from trade.

Throughout this paper, I assume that the bidders are risk neutral. A decrease in the deposit serves to shift risk from the bidders to the seller. If bidders are risk averse, then one might expect that such a shift would increase the seller's expected revenue. If this conjecture is correct, then risk aversion among the bidders would tend to make the seller's payoff decrease in the level of the deposit.

As mentioned above, the model presented here can be interpreted as a model of auctions with limited liability. All that is required is that δ be interpreted as the maximum level of losses that can be sustained by a bidder. However, such an application is confined to situations where all bidders have the same level of limited liability. Interpreted in this way, the results indicate that the seller's expected revenue is not adversely affected by the limits on the liability of the bidders. Researchers conducting auction experiments have debated the effects of limited liability on their results. [6] It is hoped that the equilibria constructed here will help to improve these assessments of the effects of limited liability on bidding behavior.

APPENDIX

Renegotiation in the Competitive Buyers Case

In the (CB) case suppose that the seller can induce the bidders to participate in a second auction following a default by the winning bidder. When the symmetric bidding strategy in the initial auction is an increasing function of a bidder's private information, then in equilibrium the private information of the bidders can be inferred from their bids. Hence, in

[6] See Kagel and Levin [6, 7], Hansen and Lott [2], and Lind and Plott [9], for analyses of the effects of limited liability on a "winner's curse" result.

equilibrium, a second auction following a default yields the same result as a complete information auction, since each bidder could infer the private information of his rivals.

Consider an auction where each bidder's value for the object at auction is known by the other bidders and ties are broken by awarding the object to the bidder with the higher value. Suppose that each bidder i's valuation for the object at auction is denoted w_i, and suppose without loss of generality that $w_1 > \cdots > w_n$.

PROPOSITION A. *Consider a first-price complete information auction. In every pure strategy equilibrium the bidder with the highest value wins and the resulting price is p such that $p \in [w_2, w_1]$.*

Proof. First I will show that any outcome where the bidder with the highest value wins and pays a price p such that $p \in [w_2, w_1]$ is supported by a pure strategy Nash equilibrium. Let b_i denote the bid of the bidder with value w_i. Notice that in a first-price auction the bids $b_1 = b_2 = p \in [w_2, w_1]$ and $b_i = w_i$ for all $i > 2$ result in the proposed outcome. (Recall that ties are broken by awarding the object at auction to the tied bidder with the highest value.) In the proposed outcome the losing bidders $2, ..., n$ attain a zero payoff. Any alternative bid that is less than or equal to p would also result in a zero payoff. Any bid above p would result in their winning the auction but they would receive a negative payoff. Hence, bidders $2, ..., n$ cannot attain a higher payoff by deviating from the bids $b_2, ..., b_n$. In the proposed outcome bidder 1 receives a nonnegative payoff. Lowering his bid would lead to a zero payoff. Raising his bid would reduce his payoff since he would still win the auction but would be required to pay a higher price. Hence, bidder 1 cannot attain a higher payoff by deviating from b_1 when the other bidders follow their proposed strategies.

Now I will show that any outcome that does not conform to the pattern described above cannot be supported by a Nash equilibrium in pure strategies. Consider an outcome where bidder 1 does not win. If the strategies resulting in this outcome are an equilibrium, then the winning bid b^* must be less than or equal to w_2; otherwise, the winning bidder would have to be earning a negative payoff. However, since $w_1 > w_2$ bidder 1 could attain a positive payoff by bidding b^* since he would win and pay a price below his value. Hence, no outcome where bidder 1 does not win could be a Nash equilibrium in pure strategies. Now consider an outcome where bidder 1 does win but $p \notin [w_2, w_1]$. If $p > w_1$, then bidder 1's payoff would be negative and he could always attain a higher payoff by lowering his bid. If $p < w_2$, then bidder 2 could submit a bid such that he wins the auction at a price below his value, thus attaining a higher payoff than an outcome where he loses. Therefore, no outcome where $p \notin [w_2, w_1]$ could be supported by a Nash equilibrium. ∎

Ruling out equilibria where bids are higher than the bidder's value, the resulting price would be equal to the second highest valuation. Hence, the seller should be able to get the winning bidder to pay at least $v(s, y)$, the second highest valuation. I am not proposing that in equilibrium the seller actually holds an auction in the renegotiation stage. If the seller does hold an auction in equilibrium, then a losing bidder who bids lower than his equilibrium bid might win in the second-stage auction. This possibility makes the payoff function nondifferentiable in the bid and rules out bidding strategies that are increasing and differentiable in the bidders private information. Suppose instead that the seller holds an auction after a default only if the winning bidder refuses to pay the price $v(s, y)$. This threat by the seller is credible, since if the bidders follow their equilibrium strategies the auction will result in a price of $v(s, y)$. Faced with this credible threat, as long as $x \geqslant y$, the winning bidder would be no worse off agreeing to pay $v(s, y)$ rather than going through with the formality of an auction.

Proof of Lemma 2. There exists a best response b^* in the interval $[\beta(\underline{x}), \beta(\bar{x})]$. Any bid less than $\beta(\underline{x})$ provides a bidder with the same expected payoff as a bid of $\beta(\underline{x})$. The probability of winning the auction is zero for any bid less than or equal to $\beta(\underline{x})$, and thus the expected payoff is zero. Similarly, any bid greater than $\beta(\bar{x})$ provides a bidder with the same expected payoff as a bid of $\beta(\bar{x})$.

If $b^* \in [\beta(\underline{x}), \beta(\bar{x})]$ is bidder 1's best response to β given $X_1 = x$, then $\beta^{-1}(b^*)$ must maximize

$$\int_{\underline{x}}^{\beta^{-1}(b^*)} \tilde{\pi}(\beta(y), x, y)\, g(y)\, dy,$$

because b^* wins if and only if $b^* > \beta(Y_1)$.

The continuity of π^{SP} and conditions (A1) and (A2) guarantee that π^{SP} is quasi-concave in z with a maximum attained at $z = x$. Hence, $\beta^{-1}(b^*) = x$.

(A3) is a necessary condition for β to be an equilibrium. If the proposed equilibrium β entails $\tilde{\pi}(\beta(\underline{x}), \underline{x}, \underline{x}) > 0$, then a bidder with private information $\underline{x}$ could do better than bidding $\beta(\underline{x})$ by increasing his bid slightly to raise his probability of winning above zero and thus raise his expected payoff above zero. Similarly, if β entails $\tilde{\pi}(\beta(\underline{x}), \underline{x}, \underline{x}) < 0$, then β cannot be an equilibrium, because the continuity of $\tilde{\pi}$ implies that there is some $x > \underline{x}$ such that $\tilde{\pi}(\beta(x), x, x) < 0$. The bidder with private information x would do better by bidding $b = \beta(\underline{x})$. ∎

Proof of Proposition 1. π^{SP} is continuous and differentiable in its first argument, and

$$\pi_1^{SP}(z,x)=\left\{\int_{s^*(\beta(z)-\delta,x,z)}^{\infty}(v(s,x)-\beta(z))\,f(s)\,ds\right.$$
$$\left.+\int_{-\infty}^{s^*(\beta(z)-\delta,x,z)}[v(s,x)-p(s,x,z)-\delta]\,f(s)\,ds\right\}g(z). \tag{4}$$

Using the fact that $b=p(s^*(b-\delta,x,y),x,y)+\delta$ and property (P2), it is easy to see from Eq. (4) that $\pi_{12}^{SP}(z,x)\geqslant 0$, for any function β, and for all $x\neq z$. In addition, Eq. (2) is equivalent to $\pi_1^{SP}(x,x)=0$ for all $x\in\bar{X}$. Therefore, (A1) and (A2) must be satisfied for any β that satisfies Eq. (2), since $\pi_1^{SP}(z,x)$ is nondecreasing in its second argument and is equal to zero when $z=x$. Condition (A3) is trivially satisfied, because Eq. (2) evaluated at $\underline{x}$ is exactly $\tilde{\pi}(\beta(\underline{x}),\underline{x},\underline{x})=0$.

Next I show that any function β that satisfies Eq. (2) is increasing. If β satisfies Eq. (2), then $b=\beta(x)$ satisfies $b=\sigma(b,x)$, where

$$\sigma(b,x)=\int_{s^*(b-\delta,x,x)}^{\infty}v(s,x)\,f(s)\,ds$$
$$+\int_{-\infty}^{s^*(b-\delta,x,x)}[v(s,x)+b-p(s,x,x)-\delta]\,f(s)\,ds.$$

The function σ is increasing in b and x. To see that $\sigma(b,x)$ is increasing in b note that

$$\sigma_1(b,x)=F(s^*(b-\delta,x,x))>0.$$

Similarly, to see that $\sigma(b,x)$ is increasing in x note that

$$\sigma_2(b,x)=\int_{s^*(b-\delta,x,x)}^{\infty}v_2(s,x)\,f(s)\,ds$$
$$+\int_{-\infty}^{s^*(b-\delta,x,x)}[v_2(s,x)-\tilde{p}_2(s,x)]\,f(s)\,ds,$$

where $\tilde{p}(s,x)=p(s,x,x)$. Under (SS) and (CB) $\tilde{p}(s,x)=v(s,x)$. Under (SB), $\tilde{p}(s,x)=E[V_0\,|\,S=s]$. Hence $\sigma_2(b,x)>0$ since $v_2(s,x)>0$ and $v_2(s,x)-\tilde{p}_2(s,x)\geqslant 0$. Now to show that β is increasing, note that $\beta(x)$ must satisfy $\sigma(\beta(x),x)-\beta'(x)=0$. The implicit function theorem implies that $\beta'(x)=\sigma_2(\beta(x),x)/[1-\sigma_1(\beta(x),x)]>0$.

It remains to be shown that a solution to Eq. (2) exists. Assumption 4 implies that $E[V_1\,|\,X_1=x]\geqslant 0$ for all $x\in\bar{X}$. It is also the case that

$\sigma(b, x) \geq E[V_1 | X_1 = x]$ for all b and x. Thus, for all x and for $b = 0$, $\sigma(b, x) - b \geq 0$. Since s^* is increasing in b, $\lim_{b \to \infty} \sigma(b, x) - b < 0$ by Assumption 6. Thus, for $\delta > 0$, there exists b' such that $\sigma(b', x) - b' \leq 0$. Therefore, $\sigma(0, x) - 0 \geq 0$ and $\sigma(b', x) - b' \leq 0$ for all $x \in \tilde{X}$. It follows that for each x there exists a $\beta(x)$ between 0 and b' such that $\sigma(\beta(x), x) - \beta(x) = 0$, since σ is continuous.

By Lemma 1 these results hold for the (NR) case. ∎

Proof of Proposition 2. The following lemmas are needed to prove the proposition.

LEMMA A. *Under assumptions* (S) *and* (MH),

$$H(x) = \int_{s^*(b-\delta, x, y)}^{\infty} v(s, x) \frac{f(s)}{1 - (s^*(b-\delta, x, y))} ds$$

is nondecreasing in x.

Proof. (S) implies that

$$H(x) = \psi(x) + E[\phi(S) \mid S \geq s^*].$$

The proof is immediate when s^* is independent of x as is the case under (CB) when $x \geq y$ and under (SB). s^* is not independent of x under (CB) when $x < y$ or under (SS). In these cases $p(s, x, y) = v(s, x)$.

Therefore, it remains to show that the lemma holds when $p(s, x, y) = v(s, x)$. Differentiating with respect to x yields

$$H'(x) = \psi'(x) + \frac{dE[\phi(S) \mid S \geq s^*]}{ds^*} \frac{\partial s^*}{\partial x}. \tag{5}$$

By the definition of s^*, when $p(s, x, y) = v(s, x)$, $\partial s^*/\partial x = -\psi'(x)/\phi'(s^*)$. Equation (5) becomes

$$H'(x) = \left[1 - \frac{dE[\phi(S) \mid S \geq s^*]}{ds^*} \frac{1}{\phi'(s^*)}\right] \psi'(x).$$

The conclusion of the lemma depends on $H'(x) \geq 0$. $dE[\phi(S) \mid S \geq s^*]/ds^* \leq \phi'(s^*)$ is sufficient for $H'(x) \geq 0$. Integrating $E[\phi(S)|S \geq s^*]$ by parts, using the assumption that $\int_s^{\infty} [1 - f(\xi)]\, d\xi < \infty$ (in (MH)) and $\lim_{s \to \infty} \phi(s)[1 - f(s)] = 0$ (in (S)), yields

$$E[\phi(S) \mid S \geq s^*] = \phi(s^*) + \frac{\int_{s^*}^{\infty} [1 - F(\xi)]\, \phi'(\xi)\, d\xi}{1 - F(s^*)}.$$

Differentiating with respect to s^* yields

$$\frac{dE[\phi(S) \mid S \geqslant s^*]}{ds^*} = \frac{\int_{s^*}^{\infty} [1 - F(\xi)]\, \phi'(\xi)\, d\xi\, f(s^*)}{[1 - F(s^*)]^2}$$

$$\leqslant \frac{\int_{s^*}^{\infty} [1 - F(\xi)]\, d\xi\, f(s^*)\, \phi'(s^*)}{[1 - F(s^*)]^2},$$

where the second inequality follows from the concavity of ϕ.

The nondecreasing hazard rate assumption (MH) is identical to the log concavity of $1 - F(s)$ (i.e., $\log[1 - F(s)]$ is concave). The log concavity of $1 - F(s)$ implies the log concavity of $\int_s^{\infty} [1 - F(\xi)]\, d\xi$ (see Karlin [8]). Furthermore, the log concavity of $\int_{s^*}^{\infty} [1 - F(\xi)]\, d\xi$ implies $\int_{s^*}^{\infty} [1 - F(\xi)]\, d\xi\, f(s^*) \leqslant [1 - F(s^*)]^2$. This implies

$$\frac{\int_{s^*}^{\infty} [1 - F(\xi)]\, d\xi\, f(s^*)\, \phi'(s^*)}{[1 - F(s^*)]^2} \leqslant \phi'(s^*),$$

and the result is proven. ∎

LEMMA B. *Suppose h and β are defined by the differential equation*

$$h(\beta(x), x)\, \mu(x) = \beta'(x),$$

with $\mu(x) > 0$, $\forall x \in \tilde{X}$. *If* $h_2(\beta(x), x) > 0$, *for all* $x \in [\underline{x}, \bar{x}]$, *and* $h(\beta(\underline{x}), \underline{x})\, \mu(\underline{x}) \geqslant 0$, *then* $\beta'(x) > 0$, $\forall x \in (\underline{x}, \bar{x}]$.

Proof. If there exists $\hat{x} \in (\underline{x}, \bar{x}]$ such that $\beta'(\hat{x}) \leqslant 0$, then $h(\beta(\hat{x}), \hat{x}) \leqslant 0$. Thus, there exists a $z \in [\underline{x}, \bar{x}]$ such that $h(\beta(z), z) = 0$ and $dh(\beta(z), z)/dz \leqslant 0$. Note that

$$\frac{dh(\beta(z), z)}{dz} = h_1(\beta(z), z)\, \beta'(z) + h_2(\beta(z), z) \leqslant 0.$$

Hence $h_2(\beta(z), z) > 0$ implies that $\beta'(z) \neq 0$. But this violates the differential equation at z. Thus, such an $\hat{x}$ cannot exist and the result is shown. ∎

To prove Proposition 2, it is sufficient to show that a solution to the differential equation β satisfying the initial condition satisfies the conditions of Lemma 3. First note that the initial condition is precisely condition (B3) of Lemma 3.

Define the function D as

$$D(z,x)=\left\{\int_{s^*(\beta(z)-\delta,x,z)}^{\infty}[v(s,x)-\beta(z)]\,f(s)\,ds\right.$$
$$\left.+\int_{-\infty}^{s^*(\beta(x)-\delta,x,z)}[v(s,x)-p(s,x,z)-\delta]\,f(s)\,ds\right\}g(z)$$
$$+\beta'(z)\int_{x}^{z}\int_{s^*(\beta(z)-\delta,x,z)}^{\infty}f(s)\,g(y)\,ds\,dy. \tag{6}$$

Notice that $D(z,x)=\pi_1^{\mathrm{FP}}(z,x)$ for $z\neq x$. Since Eq. (3) is equivalent to $D(x,x)=0$ and D is continuous in its second argument, conditions (B1) and (B2) hold, if D is nondecreasing in x. D is clearly nondecreasing in x for those cases where s^* is independent of x. (That is, for (CB) where $x\geqslant y$ and (SB).)

When s^* is not independent of x, $p(s,x,y)=v(s,x)$ and D can be written as

$$D(z,x)=\left\{\frac{\int_{s^*(\beta(z)-\delta,x,z)}^{\infty}[v(s,x)-\beta(z)]\,f(s)\,ds}{[1-F(s^*(\beta(z)-\delta,x,z))]}\right.$$
$$\left.-\delta\,\frac{F(s^*(\beta(z)-\delta,x,z))}{[1-F(s^*(\beta(z)-\delta,x,z))]}-\beta'(z)\,\frac{G(z)}{g(z)}\right\}$$
$$\times[1-F(s^*(\beta(z)-\delta,x,z))]\,g(z).$$

Here conditions (B1) and (B2) hold, if the term within the braces is nondecreasing in x. Notice that the integral term within the braces is nondecreasing by Lemma A. Under (SS), s^* is defined by $v(s^*,x)=b-\delta$. Since v is increasing in both arguments, s^* must be a decreasing function of x. Thus, the second term within the braces is increasing in x.

It remains to be shown that β is an increasing function for all $x\in\tilde{X}$. Define the function h as

$$h(b,x)=\frac{\int_{s^*(b-\delta,x,x)}^{\infty}[v(s,x)-b]\,f(s)\,ds}{\int_{\underline{x}}^{x}\int_{s^*(b-\delta,x,x)}^{\infty}f(s)\,g(y)\,ds\,dy}$$
$$+\frac{\int_{-\infty}^{s^*(\beta(x)-\delta,x,x)}[v(s,x)-p(s,x,x)-\delta]\,f(s)\,ds}{\int_{\underline{x}}^{x}\int_{s^*(b-\delta,x,x)}^{\infty}f(s)\,g(y)\,ds\,dy}.$$

Note that $h(\beta(x),x)$ is equal to the term within the braces of Eq. (3). Defining $\mu(x)=g(x)$, Eq. (3) becomes

$$h(\beta(x),x)\,\mu(x)=\beta'(x).$$

Since h is differentiable and increasing in its second argument, $h_2(\beta(x), x) > 0$, for all x. Condition (B3) implies $h(\beta(\underline{x}), \underline{x}) = 0$. This together with $\mu(x) > 0$, by Lemma B, implies that $\beta'(x) > 0$ for all $x \in (\underline{x}, \bar{x}]$, and thus β is increasing on the interval $[\underline{x}, \bar{x}]$.

By Lemma 1 these results hold for the (NR) case. ▮

Proof of Proposition 3. Let β_δ and $\beta_{\hat{\delta}}$ denote the equilibrium bidding strategies defined by the proposition. It is sufficient to show that $\delta > \hat{\delta}$ implies $\beta_\delta(x) \leqslant \beta_{\hat{\delta}}(x)$ for all $x \in \tilde{X}$. If this is true, then $s^*(\beta_\delta(x) - \delta, x, y) < s^*(\beta_{\hat{\delta}} - \hat{\delta}, x, y)$ for all $x, y \in \tilde{X}$.

Second-Price Auction Case: Eq. (2) implies that $\beta_\delta(x)$ must satisfy

$$\int_{s^*(\beta_\delta(x) - \delta, x, x)}^{\infty} v(s, x)\, f(s)\, ds + \int_{-\infty}^{s^*(\beta_\delta(x) - \delta, x, x)} [v(s, x) + \beta_\delta(x) - p(s, x, y) - \delta]\, f(s)\, ds - \beta_\delta(x) = 0.$$

The implicit function theorem implies that $d\beta_\delta / d\delta = -F(s^*)/[1 - F(s^*)] < 0$.

First-Price Auction Case: The analysis is similar to the proof of Lemma B in the proof of Proposition 2.

$$h(b, x; \delta) = \frac{\left(\begin{array}{l} \int_{s^*(b-\delta, x, x)}^{\infty} [v(s, x) - b]\, f(s)\, ds \\ \quad + \int_{-\infty}^{s^*(b-\delta, x, x)} [v(s, x) - p(s, x, x) - \delta]\, f(s)\, ds \end{array} \right)}{\int_{\underline{x}}^{x} \int_{s^*(b-\delta, x, y)}^{\infty} f(s)\, g(y)\, ds\, dy}.$$

Let $\mu(x) = g(x)$. The differential equation in Eq. (3) of Proposition 2 can be written as

$$h(\beta_\delta(x), x; \delta)\, \mu(x) = \beta'_\delta(x). \tag{7}$$

If β_δ is an increasing, symmetric, and differentiable equilibrium bidding strategy, then Eq. (7) must be satisfied.

Employing the same argument as that used above for the second-price auction, we can apply condition (B3) to conclude that $\beta_\delta(\underline{x}) < \beta_{\hat{\delta}}(x)$. Thus, if there exists an $x \in \tilde{X}$ such that $\beta_\delta(x) \geqslant \beta_{\hat{\delta}}(x)$, then there must exist a $z \in (\underline{x}, x]$ such that $\beta_\delta(z) = \beta_{\hat{\delta}}(z)$ and $\beta'_\delta(z) \geqslant \beta'_{\hat{\delta}}(z)$. By the differential equation above, this implies that $h(\beta_\delta(z), z; \delta) \geqslant h(\beta_{\hat{\delta}}(z), z; \hat{\delta})$. Differentiating $h(b, x; \delta)$ with respect to δ yields $\partial h(b, x; \delta)/\partial\delta < 0$ for all b, x, δ such that $h(b, x; \delta) > 0$. Note that $\beta'(x) > 0$ implies that the function h is positive for all relevant values of b, x, δ. However, a contradiction arises since $\delta > \hat{\delta}$ implies $h(\beta_\delta(z), z; \delta) < h(\beta_{\hat{\delta}}(z), z; \hat{\delta})$ because $\beta_\delta(z) = \beta_{\hat{\delta}}(z)$. Hence, $\beta_\delta(x) < \beta_{\hat{\delta}}(x)$ for all $x \in \tilde{X}$, and the conclusion follows. ▮

REFERENCES

1. S. BIKHCHANDANI AND J. G. RILEY, Equilibria in open common value auctions, *J. Econ. Theory* **53** (1991), 101–130.
2. R. G. HANSEN AND J. R. LOTT, The winner's curse and public information in common value auctions: Comment, *Amer. Econ. Rev.* **81** (1991), 347–361.
3. R. M. HARSTAD, Alternative common-value auction procedures: Revenue comparisons with free entry, *J. Polit. Econ.* **98** (1990), 421–429.
4. R. M. HARSTAD AND M. H. ROTHKOPF, Withdrawable bids as winner's curse insurance, *Oper. Res.*, to appear.
5. O. HART AND B. HOLMSTROM, The theory of contracts, *in* "Advances in Economic Theory–Fifth World Congress" (T. Bewley, Ed.), Cambridge Univ. Press, Cambridge, UK, 1987.
6. J. H. KAGEL AND D. LEVIN, The winner's curse and public information in common value auctions, *Amer. Econ. Rev.* **76** (1986), 894–920.
7. J. H. KAGEL AND D. LEVIN, The winner's curse and public information in common value auctions: Reply, *Amer. Econ. Rev.* **81** (1991), 362–369.
8. S. KARLIN, Some results on optimal partitioning of variance and monotonicity with truncation level, *in* "Statistics and probability: Essays in Honor of C. R. Rao" (B. Kallianpur, P. R. Krishnaiah, and J. K. Ghosh, Eds.), North-Holland/Elsevier, Amsterdam, 1982.
9. B. LIND AND C. PLOTT, The winner's curse: Experiments with buyers and with sellers, *Amer. Econ. Rev.* **81** (1991), 347–361.
10. S. MATTHEWS, Comparing auctions for risk averse buyers: A buyer's point of view, *Econometrica* **55** (1987), 633–646.
11. R. P. MCAFEE AND J. MCMILLAN, Auctions and bidding, *J. Econ. Lit.* **25** (1987), 699–738.
12. P. R. MILGROM, Rational expectations, information acquisition and competitive bidding, *Econometrica* **49** (1981), 921–943.
13. P. R. MILGROM, Auction theory, *in* "Advances in Economic Theory—Fifth World Congress" (T. Bewley, Ed.), Cambridge Univ. Press, Cambridge, 1987.
14. P. R. MILGROM AND R. J. WEBER, A theory of auctions and competitive bidding, *Econometrica* **50** (1982), 1089–1122.
15. J. G. RILEY, Ex post information in auctions, *Rev. Econ. Stud.* **55** (1988), 409–430.
16. D. F. SPULBER, Auctions and contract enforcement, *J. Law Econ. Organ.* **6** (1990), 325–345.